The statistical dynamics of homogeneous turbulence

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The steady distribution function for homogeneous turbulence is studied starting from Liouville's equation, modified by the introduction of an instantaneously fluctuating external force, which acts as a random source of energy. A new technique for solving Liouville's equation is presented giving a systematic development of the concepts of turbulent diffusion and turbulent viscosity. It amounts to a consistent generalization of the random phase approximation. When the rate of input of energy into the **k**th Fourier component $u_{\mathbf{k}}$ has a power form $h|\mathbf{k}|^{-\alpha}$, the functional form of the mean value $\langle u_{\mathbf{k}} u_{-\mathbf{k}}\rangle$ can be determined exactly in the limit of large Reynolds number; it is $Ah^{\frac{2}{3}}|\mathbf{k}|^{-\frac{1}{3}(5+2\alpha)}$. Liouville's equation proves an inadequate basis for the steady time-dependent mean $\langle u_{\mathbf{k}}(t)\, u_{-\mathbf{k}}(t')\rangle$ and a more general equation is derived. The new equation can be solved in a similar way and shows that the time-dependent correlation starts like a Gaussian in time, then passes through an exponentially decaying state, then eventually has a power dependence $|t-t'|^{-\gamma|\mathbf{k}|}$.

1. Introduction

The problem of understanding the statistical dynamics of turbulence is a difficult one for many reasons. It is reasonable therefore to study the problem under the simplest non-trivial conditions and inquire whether if, under any physically possible conditions, solutions describing the statistical distribution of fluid velocities of a turbulent system can be obtained, even if by 'physically possible' one may mean situations which, though conceivable, are not obtainable in a laboratory. In this paper the problem of steady homogeneous isotropic turbulence will be studied under such idealized conditions, allowing the exact form of the correlation functions to be determined, and thereby it is hoped to provide a foundation for the study of more realistic cases. The physical situation in an ideal turbulent fluid sounds quite straightforward. There is some mechanism by which the energy enters the system, say by the effect of a fluctuating force $\mathscr{F}(\mathbf{r},t)$. This energy then spreads amongst all the degrees of freedom of the system via the non-linear equations of motion and is eventually lost through viscosity. If the mechanism of input is postulated to be statistically defined, then it follows automatically that the rest of the system is also only defined statistically, and the problem can be reduced to the solution of differential equations in terms of the (infinite) number of degrees of freedom of the system. A similar situation arises in the kinetic theory of gases, where the solution to every problem is a solution of Liouville's equation. But from Liouville's equation there needs to be two further steps taken before a practical solution to any problem can be obtained: first, the Boltzmann or Fokker–Planck equation has to be derived, and then this equation has to be solved. Something of an analogous programme will be attempted in this paper but the actual procedures will eventually differ very considerably from those of conventional statistical dynamics since almost without exception the approximations employed in kinetic theory fail as do the methods presently employed in the quantum theory of many particles and the quantum theory of fields. Liouville's equation, and its generalization when a random input of energy is present, turns out to be a useful starting point for the derivation of the steady distribution function of velocities, but it is inadequate for the study of the correlation functions when the velocities are taken at different points in time and a more general equation is derived to handle this problem. After an intuitive discussion of the problem a general method is given for solving both Liouville's equation and the new extended equation. The solutions are then discussed in detail and it is shown that for a certain class of input behaviours exact solutions can be given for the structure of the velocity correlation function. The paper concludes with a discussion of the validity of the method of solution.

2. Liouville's equation

The simplest situation resulting in turbulence is for some random force to act on the fluid, the force being defined in the simplest possible statistical way. The simplest problem is to find the value of the velocity correlation functions of the steady homogeneous turbulence which accompanies the action of such a force, all velocities being measured at the same time. This problem is set up in this section by deriving a differential equation which will describe a fluid excited by a random force.

Consider an incompressible fluid of unit density occupying a large volume L^3. Let an external force $\mathscr{F}(\mathbf{r}, t)$ act upon it so that the Navier–Stokes equations for the velocity are

$$\frac{\partial \mathbf{U}}{\partial t} = \nu\nabla^2\mathbf{U} - (\mathbf{U}.\boldsymbol{\nabla})\,\mathbf{U} - \boldsymbol{\nabla}p + \mathscr{F}, \tag{2.1}$$

and the incompressibility condition is

$$\boldsymbol{\nabla}.\mathbf{U} = 0. \tag{2.2}$$

It is convenient to consider the Fourier components of the velocity as variables, so writing

$$\mathbf{U}_{\mathbf{k}}(t) = \int \mathbf{U}(\mathbf{r}, t)\, e^{i\mathbf{k}.\mathbf{r}}\, d^3r, \tag{2.3}$$

one has

$$\mathbf{U}(\mathbf{r}, t) = L^{-3} \sum_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{r}}\, \mathbf{U}_{\mathbf{k}}(t), \tag{2.4}$$

where $\mathbf{k}$ runs over the values $2\pi L^{-1}(n_1, n_2, n_3)$, the n being integers. For an infinite system this goes over into

$$\mathbf{U}(\mathbf{r}, t) = (2\pi)^{-3} \int \mathbf{U}_{\mathbf{k}}(t)\, e^{-i\mathbf{k}.\mathbf{r}}\, d^3k. \tag{2.5}$$

Cyclic boundary conditions have been adopted so that the (complex) variables $\mathbf{U}_\mathbf{k}$ and $\mathbf{U}_{-\mathbf{k}}$ can be employed, they being more convenient than the real variables. It is useful to make the transition to the infinite system at will, so to this end define

$$\Delta = (2\pi/L)^3$$

so that

$$\Delta \sum_\mathbf{k} \to \int d^3k.$$

Using (2.2) one may eliminate the pressure from (2.1) and obtain

$$\frac{\partial \mathbf{U}_\mathbf{k}}{\partial t} = -\nu \mathbf{k}^2 \mathbf{U}_\mathbf{k} + \frac{i\Delta}{(2\pi)^3} \sum_\mathbf{j} \{(\mathbf{U}_{\mathbf{k}-\mathbf{j}} \cdot \mathbf{j})\, \mathbf{U}_\mathbf{j} - \mathbf{k}\mathbf{k}^{-2}(\mathbf{k} \cdot \mathbf{U}_\mathbf{j})\,(\mathbf{j} \cdot \mathbf{U}_{\mathbf{k}-\mathbf{j}})\} + \mathscr{F}_\mathbf{k} - \mathbf{k}\mathbf{k}^{-2}(\mathbf{k} \cdot \mathscr{F}_\mathbf{k}), \tag{2.6}$$

$$\mathbf{k} \cdot \mathbf{U}_\mathbf{k} = 0, \tag{2.7}$$

or, writing in Cartesians,

$$\frac{\partial U^\alpha_\mathbf{k}}{\partial t} = -\nu \mathbf{k}^2 U^\alpha_\mathbf{k} + \sum_{j,l;\beta,\gamma} M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}}\, U^\gamma_\mathbf{j}\, U^\beta_\mathbf{l} + \sum_{\beta,\gamma} \mathscr{F}^\beta_\mathbf{k} \mathscr{D}^{\alpha\beta}_\mathbf{k}, \tag{2.8}$$

where

$$\mathscr{D}^{\alpha\beta}_\mathbf{k} = \delta^{\alpha\beta} - k^\alpha k^\beta / \mathbf{k}^2, \tag{2.9}$$

and

$$M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}} = \frac{i\Delta}{(2\pi)^3} (\delta^{\alpha\beta} j^\gamma - k^\alpha k^\beta j^\gamma \mathbf{k}^{-2})\, \delta_{-\mathbf{k}\mathbf{j}\mathbf{l}}. \tag{2.10}$$

The matrix M can be written in a more symmetric form with the aid of

$$M^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{j}\mathbf{l}} = \frac{\Delta}{(2\pi)^3 i} (k^\beta \mathscr{D}^{\alpha\gamma}_\mathbf{k} + k^\gamma \mathscr{D}^{\alpha\beta}_\mathbf{k})\, \delta_{\mathbf{k}\mathbf{j}\mathbf{l}}. \tag{2.11}$$

The symbol $\delta_{\mathbf{k}\mathbf{j}\mathbf{l}}$ is unity when $\mathbf{k}+\mathbf{j}+\mathbf{l} = 0$ and zero otherwise. For an infinite system it goes over into a Dirac δ function

$$\delta_{\mathbf{k}\mathbf{j}\mathbf{l}} \to \Delta\delta(\mathbf{k}+\mathbf{j}+\mathbf{l}). \tag{2.12}$$

The mean velocity $\mathbf{U}_0$ is zero, so that (2.8) represents a set of variables $\mathbf{U}_\mathbf{k}$ coupled non-linearly to one another by the term M, the non-linear term in the equation for $\mathbf{U}_\mathbf{k}$ not containing $\mathbf{U}_\mathbf{k}$ itself.

To obtain Liouville's equation one must introduce the function

$$F(\ldots, \mathbf{u}_\mathbf{k}, \ldots; t)$$

which gives the probability that the $\mathbf{U}_\mathbf{k}$ have the values $\mathbf{u}_\mathbf{k}$ at the time t. For any particular system this must be a δ function for each $\mathbf{u}_\mathbf{k}$

$$F = \prod_\mathbf{k} \delta[\mathbf{u}_\mathbf{k} - \mathbf{U}_\mathbf{k}(t)], \tag{2.13}$$

where $\mathbf{U}_\mathbf{k}(t)$ is the solution of (2.8) for the particular system being considered. Differentiating with respect to time

$$\frac{\partial F}{\partial t} = \sum_\mathbf{k} \frac{\partial \mathbf{U}_\mathbf{k}}{\partial t} \frac{\partial}{\partial \mathbf{U}_\mathbf{k}} \prod_\mathbf{j} \delta(\mathbf{u}_\mathbf{j} - \mathbf{U}_\mathbf{j}(t)). \tag{2.14}$$

A straightforward manipulation now gives Liouville's equation (Hopf 1952)

$$\frac{\partial F}{\partial t} = -\sum_{k,\alpha} \frac{\partial}{\partial u^\alpha_\mathbf{k}} \Big(-\nu \mathbf{k}^2 u^\alpha_\mathbf{k} + \sum_{j,l,\alpha,\beta} M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}}\, u^\beta_\mathbf{j} u^\gamma_\mathbf{l} + \sum_\beta \mathscr{D}^{\alpha\beta}_\mathbf{k} \mathscr{F}^\beta_\mathbf{k}\Big) F. \tag{2.15}$$

This can alternatively be written in configuration space in terms of functional derivatives

$$\frac{\partial F}{\partial t} = -\int d^3r \frac{\delta}{\delta \mathbf{u}(r)} \Big(\nu\nabla^2\mathbf{u} + (\mathbf{u}\,.\,\boldsymbol{\nabla})\,\mathbf{u} - \boldsymbol{\nabla}\int (4\pi)^{-1}\,|\mathbf{r}-\mathbf{r}'|^{-1}\,\boldsymbol{\nabla}'(\mathbf{u}'\,.\,\boldsymbol{\nabla}')\,\mathbf{u}'d^3r' + \mathscr{F} - (4\pi)^{-1}\boldsymbol{\nabla}\int |\mathbf{r}-\mathbf{r}'|^{-1}\boldsymbol{\nabla}'\mathscr{F}'d^3r' \Big) F. \quad (2.16)$$

This equation is linear in F so that, in the usual way, an ensemble average will satisfy the same equation but will be described by a smooth function F.

To specify the problem completely one must now specify the input force $\mathscr{F}_{\mathbf{k}}$. The simplest way is to take it to be a random function of time so that the probability that, over a period of time T, it is found to have the value $\mathscr{F}_{\mathbf{k}}(t)$ is

$$\mathscr{P}([\mathscr{F}_{\mathbf{k}}]) = \mathscr{N}\exp\Big(-\sum_{\mathbf{k}} \Delta \int_0^T\int_0^T \mathscr{F}_{\mathbf{k}}(t)\, g_{\mathbf{k}}^{-1}(t-t')\,\mathscr{F}_{\mathbf{k}}(t')\,dt\,dt' \Big), \quad (2.17)$$

where $\mathscr{N}$ is the appropriate normalization, and g^{-1} is the functional inverse of the correlation function $g_{\mathbf{k}}$

$$\int g_{\mathbf{k}}(t-\tau)\, g_{\mathbf{k}}^{-1}(\tau-t')\,d\tau = \delta(t-t'), \quad (2.18)$$

$$\langle \mathscr{F}_{\mathbf{k}}(t)\,\mathscr{F}_{-\mathbf{k}}(t')\rangle = \mathscr{N}\int \mathscr{F}_{\mathbf{k}}(t)\,\mathscr{F}_{-\mathbf{k}}(t')\,\mathscr{P}([\mathscr{F}])\,\delta\mathscr{F} = g_{\mathbf{k}}(t-t')\,\Delta^{-1}. \quad (2.19)$$

The symbol $\delta\mathscr{F}$ implies integration over each $\mathscr{F}_{\mathbf{k}}(t)$ at each time. (For a discussion of integrals of this type see Gel'fand & Yaglom 1960). This distribution implies that the mean of several $\mathscr{F}$ is given by

$$\langle \mathscr{F}_{\mathbf{k}_1}(t_1)\,\mathscr{F}_{\mathbf{k}_2}(t_2)\ldots\mathscr{F}_{\mathbf{k}_{2n}}(t_{2n})\rangle = \Sigma g(\tau_1-\tau_2)\,g(\tau_3-\tau_4)\ldots g(\tau_{2n-1}-\tau_{2n}) \times \delta_{\kappa_1+\kappa_2}\,\delta_{\kappa_3+\kappa_4}\ldots\delta_{\kappa_{2n-1}+\kappa_{2n}},$$

where $\tau_1\ldots\tau_{2n}$ is a permutation of $t_1\ldots t_{2n}$, $\mathbf{k}_i$ being the $\mathbf{k}$ appropriate to τ_i, and the sum is over all permutations. Physically if the force $\mathscr{F}$ is to cause turbulence one expects $g_{\mathbf{k}}(t-t')$ to decrease much faster in time than the corresponding correlation function of the velocities $\langle \mathbf{U}_{\mathbf{k}}(t)\,\mathbf{U}_{-\mathbf{k}}(t')\rangle$ so it is reasonable to specialize $g_{\mathbf{k}}$ to the very convenient form of instantaneous fluctuations

$$g_{\mathbf{k}}(t-t') = h_{\mathbf{k}}\,\delta(t-t'). \quad (2.20)$$

In this case it can be shown (Appendix 2) that if the mean distribution function averaged over the fluctuating force is called $\langle F\rangle$, i.e.

$$\langle F\rangle = \int F\mathscr{P}([\mathscr{F}])\,\delta\mathscr{F}, \quad (2.21)$$

then

$$\Big\{\frac{\partial}{\partial t} - \sum_{\mathbf{k},\alpha,\beta}\frac{\partial}{\partial u^{\alpha}_{\mathbf{k}}}\Big(\Delta^{-1}\sum_{\alpha'}\mathscr{D}^{\alpha\alpha'}_{\mathbf{k}}\,h_{\mathbf{k}}\frac{\partial}{\partial \mathbf{u}^{\alpha}_{\mathbf{k}}} + \nu\mathbf{k}^2u^{\alpha}_{\mathbf{k}} - \sum_{\beta\gamma,\mathbf{jl}} M^{\alpha\beta\gamma}_{-\mathbf{kjl}}\,u^{\beta}_{\mathbf{j}}u^{\gamma}_{\mathbf{l}}\Big)\Big\}\langle F\rangle = 0. \quad (2.22)$$

Since the mean rate at which energy enters the system

$$\frac{\partial}{\partial t}\int \Delta \tfrac{1}{2} \sum_{\mathbf{k}} \mathbf{u}_{\mathbf{k}} . \mathbf{u}_{-\mathbf{k}} \langle F \rangle \prod_{\mathbf{j}} du_{\mathbf{j}} \tag{2.23}$$

is now fixed and equals $\int h_{\mathbf{k}} d^3k$ there will exist a solution of (2.22) which will be time independent and correspond to the steady mean state of the turbulent fluid. Thus the probability of finding a velocity field $\mathbf{u}_{\mathbf{k}}$ in the steady state with random input of $h_{\mathbf{k}}$ per mode is given by the solution of

$$\sum_{\mathbf{k},\,\alpha} \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}} \left(\Delta^{-1} \sum_{\alpha'} \mathscr{D}_{\mathbf{k}}^{\alpha\alpha'} h_{\mathbf{k}} \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}} + \nu \mathbf{k}^2 \mathbf{u}_{\mathbf{k}}^{\alpha} - \sum_{\beta\gamma,\,\mathbf{jl}} M_{-\mathbf{k}\mathbf{j}\mathbf{l}}^{\alpha\beta\gamma} u_{\mathbf{j}}^{\beta} u_{\mathbf{l}}^{\gamma} \right) \langle F \rangle = 0. \tag{2.24}$$

This equation is a good starting point to discuss $\langle F \rangle$ but one also needs the mean values of quantities like $U_{\mathbf{k}}^{\alpha}(t)\, U_{\mathbf{k}}^{\alpha'}(t')$ and $U_{\mathbf{k}}^{\alpha}(t_1)\, U_{\mathbf{j}}^{\beta}(t_2)\, U_{\mathbf{l}}^{\gamma}(t_3)$. (It will be understood that the words 'time dependence' in this paper will always mean the dependence of such quantities as $\langle U_{\mathbf{k}}^{\alpha}(t)\, U_{-\mathbf{k}}^{\alpha'}(t') \rangle$ upon $t-t'$. The problem of the decay from a turbulent to a quiescent system will not be considered here.) The averaged Liouville equation in principle contains this information in its Green function G which satisfies

$$\left(\frac{\partial}{\partial t} - \mathscr{L} \right) G = \prod_{\mathbf{k}} \delta(u_{\mathbf{k}} - u'_{\mathbf{k}})\, \delta(t-t'), \tag{2.25}$$

$\mathscr{L}$ being the operator in equation (2.22). But in practice this is not a useful starting point and a more general approach will now be given.

3. The distribution function in generalized phase space

The Liouville equation is a useful starting point in the kinetic theory of gases for one can further approximate (2.22) or (2.25) into a Fokker–Planck or Boltzmann equation, due to the fact that one is able to distinguish the time taken between collisions and the time taken during a collision, the latter being much shorter than the former. It is this feature which makes the separation of the time from all the other variables such a useful feature of Liouville's equation. In turbulence, however, it will be shown (and is indeed clear) that no such separation can take place and one needs an equation in which time is treated on the same footing as the space variables, in order to handle this or indeed any problem in which collective behaviour is important. The distinction between a method which describes the evolution of a system in time, and a method which discusses the entire history of the system is the distinction between Hamiltonian and Lagrangian formulations of classical dynamics. Liouville's equation gives a phase space description following the Hamiltonian point of view (even though there may be no Hamiltonian function as is the case here). So what is required here is a method based on Lagrangian statistical mechanics. This is deceptively easy to write down, but has to be developed somewhat before it reaches the useful form (3.5) below.

It is convenient to use the four-dimensional Fourier transform

$$\mathbf{U}_{\mathbf{k}k_0} = \int \mathbf{U}(\mathbf{r}, t) \exp(i\mathbf{k} . \mathbf{r} + ik_0 t)\, d^3r\, dt. \tag{3.1}$$

Then equation (2.8) becomes

$$-(ik_0+\nu\mathbf{k}^2)\,U^\alpha_k+\Sigma\,M^{\alpha\beta\gamma}_{-kjl}\,U^\beta_j\,U^\gamma_l+\sum_\beta \mathscr{F}^\beta_k\mathscr{D}^{\alpha\beta}_{\mathbf{k}}=0, \tag{3.2}$$

where the (italic) k, j, l now stand for $(\mathbf{k}, k_0)$ and so on; in future this notation will always be used. In analogy with (2.13) one may introduce the probability of $\mathbf{U}_k$ taking the value $\mathbf{u}_k$, i.e. the probability, not of finding the system in a particular state at a particular time, but the probability of finding the entire history and future of the system. For a definite system this has a δ function form

$$P=\prod_k\,\delta(\mathbf{u}_k-\mathbf{U}_k). \tag{3.3}$$

Now since $\mathbf{U}_k$ satisfies (3.2) it follows that if the whole of the left-hand side of (3.2) is denoted by X_k

$$X_kP=0\quad(\text{all}\,k). \tag{3.4}$$

This is then the Lagrangian description of the system. It clearly implies that P is a δ function of the equations of motion (3.2), and is rather too general a starting point. One expects the system to fill phase space and will be content if for example all the moments of (3.4) are satisfied rather than the continuous infinity of equations which constitute (3.4). In other words one would like *one* equation which for a statistical system will have the appropriate solution of (3.4) as its solution. To get one equation, (3.4) must be multiplied by some operator which is a function of k and integrated with respect to k, and the resulting single functional equation should be capable of reproducing the moments of (3.4) obtained by multiplying by $u_pu_q\ldots$ and averaging over all u. Clearly the only operator which will fulfil this role is $\partial/\partial\mathbf{u}_k$, and the resulting equation is

$$\int dk_0\sum_{\mathbf{k},\alpha}\frac{\partial}{\partial u^\alpha_k}\{(ik_0+\nu\mathbf{k}^2)\,u^\alpha_k-\Sigma\,M^{\alpha\beta\gamma}_{-kjl}u^\beta_ju^\gamma_l-\sum_\beta\mathscr{D}^{\alpha\beta}_{\mathbf{k}}\mathscr{F}^\beta_k\}\,P=0. \tag{3.5}$$

There will be solutions of this equation which will not be solutions of (3.4) but the mode of solution presented below will always guarantee that the solution obtained is the correct one. By multiplying by u^ϵ_p (or indeed any function of the u) and integrating over all u, for a definite system (i.e. one with P given by (3.3)) it leads back to (3.2). But since it is linear it holds equally for ensemble averages. The term ensemble average must not be taken literally, however, since it no longer has the same meaning as 'time average' as it does in normal statistical mechanics. If the force is distributed according to (2.17) which is now to be written as

$$\mathscr{P}([\mathscr{F}])=\mathscr{N}\exp\left(-\sum_{\mathbf{k}}\int dk_0\,\mathscr{F}_k\mathscr{F}_{-k}\Delta/g\right), \tag{3.6}$$

then the mean P can be written down at once as a functional by substituting from (3.2) into (3.5). This gives

$$\langle P\rangle=\mathscr{N}\exp\left(-\Delta\sum_{\alpha,\mathbf{k}}\int dk_0\{[(ik_0+\nu k^2)\,u^\alpha_k-\sum_{\beta\gamma,jl}M^{\alpha\beta\gamma}_{-kjl}\,u^\beta_j\,u^\gamma_l]\right.$$
$$\left.\times\,g_k^{-1}[(-ik_0+\nu k^2)\,u^\alpha_k-\sum_{j'l',\beta'\gamma'}M^{\alpha\beta'\gamma'}_{k'j'l'}\,u^{\beta'}_{j'}\,u^{\gamma'}_{l'}]\}\right), \tag{3.7}$$

where the symbol $\mathscr{N}$ is always used for the appropriate normalization. This gives an illustration of what to expect for the mean solution of (3.4). Though in principle it is a solution to the problem of turbulence, as it stands it is quite useless in practice. The task now to find a method of deriving $\langle F\rangle$ and $\langle P\rangle$ in a useful form and to this end a simple model will now be discussed in detail which will suggest the right approach.

4. A simple model

The non-linear term ΣMuu causes the turbulence and the simplest way to think of it is as a force which as far as one is concerned is roughly random. This suggests examining the problem of the response of a linear system to a random force in detail and though all the results of this section are well known it is useful to be reminded of them and to present them in an appropriate way. Consider then the motion governed by

$$\frac{\partial U}{\partial t} = -JU+\mathscr{F}, \tag{4.1}$$

where $\mathscr{F}$ is statistically specified as in (2.17). (The label $\mathbf{k}$ and the vector character of U are dropped in the model; $J > 0$.) Then Liouville's equation is

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}(Ju-\mathscr{F})\right)F = 0, \tag{4.2}$$

or, more usefully, the Green function of Liouville's equation satisfies

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}(Ju-\mathscr{F})\right)G(u,u';t,t') = \delta(u-u')\,\delta(t-t'). \tag{4.3}$$

This function has the property that it propagates F, i.e. if $F(u,t')$ is the value of F at the time t', then at a subsequent time t

$$F(u,t) = \int G(u,u';\, t,t')\,F(u',t')\,du'. \tag{4.4}$$

Now if one starts at a time t' with a definite $F(u,t')$ and the force $\mathscr{F}$ has a distribution (defined from t' on) of $\mathscr{P}([\mathscr{F}])$ then the average of F at t, $\langle F(u,t)\rangle$ say, is propagated by the average of G, $\langle G\rangle$ say, defined by

$$\langle G\rangle = \int \mathscr{P}([\mathscr{F}])\,G([\mathscr{F}])\,\delta\mathscr{F}. \tag{4.5}$$

This follows by multiplying (4.4) by $\mathscr{P}$ and integrating over the function $\mathscr{F}$. The mean Green function is well known in the theory of Brownian motion so the result of the integration (4.5) will be quoted. (A derivation is given in Appendix 1 since its generalization is needed in Appendix 2.) It is

$$\langle G\rangle = \left(\frac{I}{\pi}\right)^{\frac{1}{2}} \exp\left[-(u-u'e^{-J(t-t')})^2/I(t,t')\right]\Theta(t-t'), \tag{4.6}$$

where $$I(t,t') = \frac{1}{2}\int_{t'}^{t}\int_{t'}^{t}\exp\left[-J(t-\tau_1) - J(t-\tau_2)\right] g(\tau_1-\tau_2)\, d\tau_1\, d\tau_2, \tag{4.7}$$

$$\Theta(t-t') = 1 \quad (t > t'),$$
$$= 0 \quad (t < t'),$$

and g is defined by (2.19). Two properties of I need to be noted. First

$$I(t,t) = 0$$

and secondly

$$\begin{aligned}\lim_{t-t'\to\infty} I(t,t') &= \text{const.}\\ &= \frac{1}{2}\int_0^{\infty}\int_0^{\infty}\exp\left[-J(\tau_1+\tau_2)\right] g(\tau_1-\tau_2)\, d\tau_1\, d\tau_2\\ &= \tfrac{1}{2}\bar{g} \quad \text{say.}\end{aligned} \tag{4.8}$$

Thus $\langle G\rangle$ loses its dependence upon u' as $t-t'$ tends to infinity and this implies that whatever $F(u,t')$ is, $\langle F\rangle$ settles down to the Gaussian

$$\langle F\rangle \to \left(\frac{\bar{g}}{2\pi}\right)^{\frac{1}{2}} \exp\left(-u^2/2\bar{g}\right). \tag{4.9}$$

In particular, if g has an exponential form

$$g = \gamma e^{-\varpi t}, \tag{4.10}$$

then

$$I(t,t') = \tfrac{1}{2}\gamma\{J^{-1}(J+\varpi)^{-1} - (J^2+\varpi^2)^{-1}\exp\left[-(J+\omega)(t-t')\right] + J^{-1}(J+\varpi)^{-1}\exp - 2J(t-t')]\}, \tag{4.11}$$

and $$\bar{g} = J^{-1}(J+\varpi)^{-1}\gamma. \tag{4.12}$$

Since $\langle G\rangle$ propagates $\langle F\rangle$ the equations for $\langle F\rangle$ and $\langle G\rangle$ can be written down from the exact expression (4.6)

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\left(Ju + \left\{\frac{1}{2}\frac{\partial I}{\partial t} + JI\right\}\frac{\partial}{\partial u}\right)\right]\langle G\rangle = \delta(u-u')\,\delta(t-t'), \tag{4.13}$$

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\left(Ju + \left\{\frac{1}{2}\frac{\partial I}{\partial t} + JI\right\}\frac{\partial}{\partial u}\right)\right]\langle F\rangle = 0. \tag{4.14}$$

It follows that as $t-t'$ tends to infinity, the steady value of $\langle F\rangle$ satisfies

$$\left[\frac{\partial}{\partial u}\left(\gamma(J+\varpi)^{-1}\frac{\partial}{\partial u} + Ju\right)\right]\langle F\rangle = 0, \tag{4.15}$$

which has the solution (4.9), i.e.

$$\langle F\rangle = \left\{\frac{2\gamma\pi}{J(J+\varpi)}\right\}^{\frac{1}{2}} \exp\left[-u^2 J(J+\varpi)/2\gamma\right], \tag{4.16}$$

while $\langle G\rangle$ settles down to the solution of

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\left(-Ju - \gamma(J+\varpi)^{-1}\frac{\partial}{\partial u}\right)\right]\langle G\rangle = \delta(u-u')\,\delta(t-t'). \tag{4.17}$$

A straightforward generalization is to the case of several $\mathscr{F}$ characterized by constants γ_α, ϖ_α. Then (4.17) has the factor $\gamma(J+\varpi)^{-1}$ replaced by

$$\sum_\alpha \gamma_\alpha(J+\varpi_\alpha)^{-1}. \tag{4.18}$$

Another way of stating these results, which will turn out to be the basis for subsequent work, is to observe that if the equation for G is written

$$\left[\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\left(K\frac{\partial}{\partial u}+Ju\right)+K\frac{\partial^2}{\partial u^2}-\mathscr{F}\frac{\partial}{\partial u}\right]G = \delta(u-u')\,\delta(t-t'), \tag{4.19}$$

where

$$K = \tfrac{1}{2}\partial I/\partial t+JI,$$

and this is expanded in terms of the solution of (4.13), which for the moment will be called Γ rather than $\langle G\rangle$, then

$$G = \Gamma-\left\{\int\Gamma\mathscr{F}\frac{\partial}{\partial u}\Gamma\right\}+\left\{\int\int\Gamma\mathscr{F}\frac{\partial}{\partial u}\Gamma\mathscr{F}\frac{\partial}{\partial u'}\Gamma-\int\Gamma K\,\partial^2\Gamma/\partial u^2\right\}+\ldots. \tag{4.20}$$

In this expansion terms of order γ^n and of order $\mathscr{F}^{2n}$ (and mixed terms like $\gamma^{n-m}\mathscr{F}^{2n-2m}$) are collected together. Upon averaging the expression in each brace vanishes, as do all terms odd in $\mathscr{F}$, and $\langle G\rangle = \Gamma$.

Clearly the second-order brace can be considered as defining γ,

$$\lim_{t-t'\to\infty}\left\langle\int\int\Gamma\mathscr{F}\frac{\partial}{\partial u}\Gamma\mathscr{F}\frac{\partial}{\partial u'}\Gamma\right\rangle = \int\Gamma\gamma(J+\varpi)^{-1}\,\partial^2\Gamma/\partial u^2, \tag{4.21}$$

and hence Γ, i.e. $\langle G\rangle$ itself. This of course is just what one would expect from a Gaussian distribution: it is specified by its second moment, which in this case is then related to the externally given γ, ϖ, and J. A straightforward extension of this property will now be developed to get an intuitive solution of (2.24).

5. A simple derivative of the turbulent distribution function

In this section the model discussed in §4 will be used to derive the equations for the mean distribution function, using an intuitive argument. Consider then the equation

$$\left[\frac{\partial}{\partial t}-\sum_{\alpha;\,\mathbf{k}}\frac{\partial}{\partial u_{\mathbf{k}}^\alpha}\left(\frac{\partial}{\partial u_{\mathbf{k}}^\beta}h_{\mathbf{k}}^\alpha\mathscr{D}_{\mathbf{k}}^{\alpha\beta}\Delta^{-1}+\nu\mathbf{k}^2u_{\mathbf{k}}^\alpha-\sum_{\mathbf{j},\mathbf{l};\,\beta\gamma}M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}}u_{\mathbf{j}}^\beta u_{\mathbf{l}}^\gamma\right)\right]$$
$$\times\, G(\ldots\mathbf{u}_{\mathbf{k}},\mathbf{u}'_{\mathbf{k}}\ldots;\,t,t') = \Pi\delta(\mathbf{u}_{\mathbf{k}}-\mathbf{u}'_{\mathbf{k}})\,\delta(t-t'). \tag{5.1}$$

(The mean value signs $\langle\ldots\rangle$ will now be dropped since these alone are referred to when discussing F and G.) This equation describes the physical situation of energy entering the system due to the fluctuating forces, at a rate $h_{\mathbf{k}}$, and leaving due to viscosity $\nu\mathbf{k}^2$. The term ΣMuu neither creates nor destroys energy but mixes it up amongst the various $\mathbf{u}_{\mathbf{k}}$. To some extent, as far as one particular $\mathbf{u}_{\mathbf{k}}$ is concerned, ΣMuu must appear as a random force of the type discussed in §4. But in addition the force acting upon $\mathbf{u}_{\mathbf{j}}$ say will contain a term $Mu_{\mathbf{k}}u_{-\mathbf{k}+\mathbf{j}}$ so that the gain of energy by the component $\mathbf{u}_{\mathbf{j}}$ will depend on the magnitude of

$\mathbf{u_k}$ and hence the loss of energy by $\mathbf{u_k}$ will depend upon its magnitude. One can expect then that as far as one component $\mathbf{u_k}$ is concerned the effect of the rest of the turbulent fluid will be to produce a diffusive term represented as in §4 by a second derivative, and also a term which represents a dissipative force proportional to $\mathbf{u_k}$, i.e. to use the nomenclature of the Fokker–Planck equation, a dynamical friction. Thus one may expect the change of just one component $\mathbf{u_k}$ (and $\mathbf{u_{-k}}$ for they always appear together) to be described by

$$\left[\frac{\partial}{\partial t}-\sum_\alpha \frac{\partial}{\partial u_{\mathbf{k}}^\alpha}\left(\sum_\beta \mathscr{D}_{\mathbf{k}}^{\alpha\beta} d_{\mathbf{k}} \Delta^{-1}\frac{\partial}{\partial u_{-\mathbf{k}}^\beta}+\omega_{\mathbf{k}} u_{\mathbf{k}}^\alpha\right)\right.$$
$$\left.-\sum_\alpha \frac{\partial}{\partial u_{-\mathbf{k}}^\alpha}\left(\sum_\beta \mathscr{D}_{\mathbf{k}}^{\alpha\beta} d_{\mathbf{k}} \Delta^{-1}\frac{\partial}{\partial u_{\mathbf{k}}^\beta}+\omega_{\mathbf{k}} u_{-\mathbf{k}}^\alpha\right)\right] G_{\mathbf{k}}(u_{\mathbf{k}} u_{-\mathbf{k}} u'_{\mathbf{k}} u'_{-\mathbf{k}} t-t')$$
$$= \delta(\mathbf{u_k}-\mathbf{u'_k})\,\delta(\mathbf{u_{-k}}-\mathbf{u'_{-k}})\,\delta(t-t') \quad (d_{\mathbf{k}}=d_{-\mathbf{k}}, \omega_{\mathbf{k}}=\omega_{-\mathbf{k}}). \quad (5.2)$$

In the same way the steady distribution function will be governed by

$$\left\{\sum_\alpha \frac{\partial}{\partial u_{\mathbf{k}}^\alpha}\left(\sum_\beta \mathscr{D}_{\mathbf{k}}^{\alpha\beta}\Delta^{-1} d_{\mathbf{k}}\frac{\partial}{\partial u_{\mathbf{k}}^\beta}+\omega_{\mathbf{k}} u_{\mathbf{k}}^\alpha\right)\right.$$
$$\left.+\sum_\alpha \frac{\partial}{\partial u_{-\mathbf{k}}^\alpha}\left(\sum_\beta \mathscr{D}_{\mathbf{k}}^{\alpha\beta}\Delta^{-1} d_{\mathbf{k}}\frac{\partial}{\partial u_{\mathbf{k}}^\beta}+\omega_{\mathbf{k}} u_{-\mathbf{k}}^\alpha\right)\right\} f(\mathbf{u_k}, \mathbf{u_{-k}}) = 0. \quad (5.3)$$

The diffusion constant $d_{\mathbf{k}}$, and the total viscosity $\omega_{\mathbf{k}}$ can be written

$$d_{\mathbf{k}} = h_{\mathbf{k}}+S_{\mathbf{k}}, \quad (5.4)$$

$$\omega_{\mathbf{k}} = \nu\mathbf{k}^2+R_{\mathbf{k}}, \quad (5.5)$$

i.e. total input into components $\mathbf{u_k}$ (and $\mathbf{u_{-k}}$)

= external input + input from all other components; (5.4)

total output

= viscous loss (i.e. external output) + output into all other components. (5.5)

It must be emphasized that this section is only introductory to the next, so the derivation of $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ which will now follow will employ assumptions that are unnecessary and sometimes incorrect, but it will turn out that the forms of $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ so obtained are correct, so that the consequences of assuming (5.2) are better founded than (5.2) itself. With this proviso the solution of (5.3) is clearly

$$f(\mathbf{u_k},\mathbf{u_{-k}}) = (\Delta/\sqrt{2\pi q_{\mathbf{k}}})\exp\left(-\sum_\alpha u_{\mathbf{k}}^\alpha u_{-\mathbf{k}}^\alpha \Delta/q_{\mathbf{k}}\right)$$
$$= l_{\mathbf{k}}\exp\left(-\sum_\alpha u_{\mathbf{k}}^\alpha u_{-\mathbf{k}}^\alpha \Delta/q_{\mathbf{k}}\right) \quad \text{say}, \quad (5.6)$$

where
$$q_{\mathbf{k}} = d_{\mathbf{k}}/\omega_{\mathbf{k}}. \quad (5.7)$$

(There is always implicit in f the restriction $\operatorname{div}\mathbf{u} = 0$ so that the normalization is appropriate to two degrees of freedom rather than three. The $\mathscr{D}_{\mathbf{k}}^{\alpha\beta}$ in (5.2) and (5.3) ensure that the constraint is not violated by the motion, but in practice it is most convenient to keep the constraint as a subsidiary condition and in effect drop $\mathscr{D}$ from the equations.) Then the mean of $U^\alpha_{\mathbf{k}} U^\beta_{-\mathbf{k}}$ can be expressed in terms of $q_{\mathbf{k}}$,

$$\langle U^\alpha_{\mathbf{k}} U^\beta_{-\mathbf{k}} \rangle \equiv \int u^\alpha_{\mathbf{k}} u^\beta_{-\mathbf{k}} f(\mathbf{u}_{\mathbf{k}}, \mathbf{u}_{-\mathbf{k}})\, d\mathbf{u}_{\mathbf{k}} d\mathbf{u}_{-\mathbf{k}} \delta(\operatorname{div} \mathbf{u}) \tag{5.8}$$

$$= \Delta^{-1} q_{\mathbf{k}} \mathscr{D}^{\alpha\beta}_{\mathbf{k}}. \tag{5.9}$$

If (5.2) is multiplied by $u^\alpha_{\mathbf{k}}$ on the left and by $f u^\beta_{-\mathbf{k}}$ on the right one obtains upon integration over all u the relation

$$\langle U^\alpha_{\mathbf{k}}(t)\, U^\beta_{-\mathbf{k}}(t') \rangle = \int u^\alpha_{\mathbf{k}} G_{\mathbf{k}} u'^\beta_{-\mathbf{k}} f(\mathbf{u}'_{\mathbf{k}} \mathbf{u}'_{-\mathbf{k}})\, \delta(\operatorname{div} \mathbf{u})\, d\mathbf{u}_{\mathbf{k}} d\mathbf{u}_{-\mathbf{k}} d\mathbf{u}'_{\mathbf{k}} d\mathbf{u}'_{-\mathbf{k}} \tag{5.10}$$

$$= \langle U^\alpha_{\mathbf{k}}(0)\, U^\beta_{-\mathbf{k}}(0) \rangle \exp\left[-\omega_{\mathbf{k}}(t-t')\right] \quad (t > t'). \tag{5.11}$$

From these results and those of § 4 one may obtain $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$. First, consider $S_{\mathbf{k}}$ which is the analogue of the expression

$$\sum_\alpha \gamma_\alpha (J + \omega_\alpha)^{-1}.$$

The present $\omega_{\mathbf{k}}$ is the analogue of J, the lifetime associated with the one component $\mathbf{k}$, and the ϖ_α is the lifetime of the fluctuating force, i.e. of

$$\sum_{\mathbf{j},\mathbf{l};\,\beta,\gamma} M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}} U^\beta_{\mathbf{j}} U^\gamma_{\mathbf{l}}.$$

The expression γ is the analogue of the mean square of the force, i.e. of

$$\langle \sum_{\beta,\gamma;\,\mathbf{j},\mathbf{l}} M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}} u^\beta_{\mathbf{j}} u^\gamma_{\mathbf{l}} \sum_{\beta',\gamma';\,\mathbf{j}',\mathbf{l}'} M^{\alpha'\beta'\gamma'}_{\mathbf{k}\mathbf{j}'\mathbf{l}'} u^{\beta'}_{\mathbf{j}'} u^{\gamma'}_{\mathbf{l}'} \rangle. \tag{5.12}$$

This is readily evaluated if it is assumed that in the first approximation the components are independent and their distribution function

$$F = \prod_{\mathbf{j}} f(\mathbf{u}_{\mathbf{j}}, \mathbf{u}_{-\mathbf{j}}) \tag{5.13}$$

$$= \prod_{\mathbf{j}} l_{\mathbf{j}} \exp\left[-\sum_{\alpha,\mathbf{j}} u^\alpha_{\mathbf{j}} u^\alpha_{-\mathbf{j}}\, \Delta/q_{\mathbf{j}}\right]. \tag{5.14}$$

(It must be understood that terms are not counted twice in the exponent.) One then has the integral

$$\begin{aligned}\langle u^\beta_{\mathbf{j}} u^\gamma_{\mathbf{l}} u^{\beta'}_{\mathbf{j}'} u^{\gamma'}_{\mathbf{l}'} \rangle &= \int u^\beta_{\mathbf{j}} u^\gamma_{\mathbf{l}} u^{\beta'}_{\mathbf{j}'} u^{\gamma'}_{\mathbf{l}'} \exp\Big(-\sum_{\alpha,\mathbf{j}} u^\alpha_{\mathbf{j}} u^\alpha_{-\mathbf{j}} \Delta/q_{\mathbf{j}}\Big)\, \delta(\operatorname{div} \mathbf{u})\, \Pi du_{\mathbf{l}} \Pi l_{\mathbf{j}} \\ &= \mathscr{D}^{\beta\gamma}_{\mathbf{j}} \mathscr{D}^{\beta'\gamma'}_{\mathbf{j}'} \delta_{\mathbf{j}+\mathbf{l}} \delta_{\mathbf{j}'+\mathbf{l}'} q_{\mathbf{j}} q_{\mathbf{j}'}/\Delta^2 + \mathscr{D}^{\beta\beta'}_{\mathbf{j}} \mathscr{D}^{\gamma\gamma'}_{\mathbf{l}} \delta_{\mathbf{j}+\mathbf{j}'} \delta_{\mathbf{l}+\mathbf{l}'} q_{\mathbf{j}} q_{\mathbf{l}}/\Delta^2 \\ &\quad + \mathscr{D}^{\beta\gamma'}_{\mathbf{j}} \mathscr{D}^{\gamma\beta'}_{\mathbf{l}} \delta_{\mathbf{j}+\mathbf{l}'} \delta_{\mathbf{l}+\mathbf{j}'} q_{\mathbf{j}} q_{\mathbf{l}}/\Delta^2.\end{aligned}$$

(The possibility of $\mathbf{j} = \mathbf{l} = \mathbf{j}' = \mathbf{l}'$ gives a contribution smaller by a factor Δ than those quoted and is disregarded. Of the three terms the first gives zero, a property of $M^{\alpha\beta\gamma}$, and the answer stems from the remaining two which have $\mathbf{j}+\mathbf{j}' = 0$, $\mathbf{l}+\mathbf{l}' = 0$ and $\mathbf{l}+\mathbf{j}' = 0$, $\mathbf{j}+\mathbf{l}' = 0$ respectively. This then implies that the $\sum_\alpha \varpi_\alpha$ is represented by $\omega_{\mathbf{j}} + \omega_{\mathbf{l}}$. Finally then, using (5.9) for the means,

$$\mathscr{D}^{\alpha\alpha'}_{\mathbf{k}} S_{\mathbf{k}} = \sum_{\substack{\mathbf{j},\mathbf{l};\,\alpha,\beta,\gamma \\ \alpha',\beta',\gamma'}} (\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}})^{-1} M^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{j}\mathbf{l}} M^{\alpha'\beta'\gamma'}_{\mathbf{k}-\mathbf{j}-\mathbf{l}} (\mathscr{D}^{\beta\beta'}_{\mathbf{j}} \mathscr{D}^{\gamma\gamma'}_{\mathbf{l}} + \mathscr{D}^{\beta\gamma'}_{\mathbf{j}} \mathscr{D}^{\gamma\beta'}_{\mathbf{l}})\, q_{\mathbf{j}} q_{\mathbf{l}}, \tag{5.15}$$

or in the limit of infinitely large volume

$$S_{\mathbf{k}} = \int \frac{L_{\mathbf{kjl}} q_{\mathbf{j}} q_{\mathbf{l}} \, d^3 j \, d^3 l}{\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}}}. \tag{5.16}$$

The expression for L is derived in Appendix 3 from (5.15) and is most conveniently written

$$L_{\mathbf{kjl}} = (2\pi)^{-6} \delta(\mathbf{k}+\mathbf{j}+\mathbf{l}) \tfrac{1}{2}\{\mathbf{k}^2[1 - 2\cos^2\theta_{kj}\cos^2\theta_{kl} + \cos\theta_{kj}\cos\theta_{kl}\cos\theta_{lj}] \\ - (\mathbf{l}^2 + \mathbf{j}^2 + \mathbf{l}\,.\,\mathbf{j})(\cos^2\theta_{kl} - \cos^2\theta_{kj})\} \tag{5.17}$$

(where it will be noted that the second part of L is antisymmetric under the interchange of $\mathbf{l}$ and $\mathbf{j}$, so it contributes nothing to the integral for $S_{\mathbf{k}}$; but the notation is very convenient when $R_{\mathbf{k}}$ is evaluated below).

To obtain $R_{\mathbf{k}}$ one may argue this way. The coefficient can be derived by expanding the complete G about $\Pi G_{\mathbf{k}}$ and averaging the analogue of the series (4.20) on the assumption that $f \simeq \prod_{\mathbf{k}} f_{\mathbf{k}}$. Thus S_k arises from the term

$$\int \cdots \int \sum_{\alpha\beta\gamma;\,\mathbf{jl}} \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}} M_{-\mathbf{kjl}}^{\alpha\beta\gamma} u_{\mathbf{j}}^{\beta} u_{\mathbf{l}}^{\gamma} \prod_{\mathbf{p}} G_{\mathbf{p}} \sum \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha'}} M_{\mathbf{kj'l'}}^{\alpha'\beta'\gamma'} u_{\mathbf{j}'}^{\prime\beta'} u_{\mathbf{l}'}^{\prime\gamma'} \prod_{\mathbf{p}} G'_{\mathbf{p}} \Pi f' \Pi \, du'. \tag{5.18}$$

But there is another term with a non-vanishing average when all but $u_{\mathbf{k}}, u_{-\mathbf{k}}$ are averaged; it is

$$\int \cdots \int \sum_{\alpha\beta\gamma;\,\mathbf{jl}} \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}} M_{-\mathbf{kjl}}^{\alpha\beta\gamma} u_{\mathbf{j}}^{\beta} u_{\mathbf{l}}^{\gamma} \prod_{\mathbf{p}} G_{\mathbf{p}} \sum_{\alpha'\beta'\gamma';\,\mathbf{j'l'}} \frac{\partial}{\partial u_{\mathbf{l}'}^{\alpha''}} \\ \times \{M_{\mathbf{l'kj'}}^{\alpha'\beta'\gamma'} u_{\mathbf{k}}^{\prime\beta'} u_{\mathbf{j}'}^{\gamma'} + M_{\mathbf{l'j'k}}^{\alpha'\beta'\gamma'} u_{\mathbf{l}'}^{\prime\beta'} u_{\mathbf{k}}^{\prime\gamma'}\} \Pi G'_{\mathbf{p}} \Pi f' \Pi \, du'. \tag{5.19}$$

Now it is a tricky matter to average away the $u_{\mathbf{p}}$ ($\mathbf{p} \neq \mathbf{k}, -\mathbf{k}$) but the correct procedure is to consider the $\partial/\partial u'_{l'}$ acting upon f'_l giving

$$-\iint \sum_{\alpha\beta\gamma;\,\mathbf{jl}} \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}} M_{-\mathbf{kjl}}^{\alpha\beta\gamma} u_{\mathbf{j}}^{\beta} u_{\mathbf{l}}^{\gamma} \prod_{\mathbf{p}} G_{\mathbf{p}} \sum_{\alpha'\beta'\gamma';\,\mathbf{l'j'}} \frac{u_{-\mathbf{l}}^{\prime\alpha'} \Delta}{q_{l'}} \\ \times (M_{\mathbf{l'kj'}}^{\alpha'\beta'\gamma'} u_{\mathbf{k}}^{\prime\beta'} u_{\mathbf{j}}^{\prime\gamma'} + M_{\mathbf{l'j'k}}^{\alpha'\beta'\gamma'} u_{\mathbf{l}'}^{\prime\beta'} u_{\mathbf{k}}^{\prime\gamma'}) \prod_{\mathbf{p}} G'_{\mathbf{p}} \Pi f' \Pi \, du'. \tag{5.20}$$

Just as with $S_{\mathbf{k}}$ this now gives $R_{\mathbf{k}}(\partial/\partial u_{\mathbf{k}}^{\alpha})\, u_{\mathbf{k}}^{\alpha}$, where

$$R_{\mathbf{k}} = \int \frac{L_{\mathbf{ljk}} q_{\mathbf{j}} \, d^3 j}{\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}}} \tag{5.21}$$

and

$$L_{\mathbf{ljk}} = M_{-\mathbf{kjl}}^{\alpha\beta\gamma} (M_{\mathbf{lkj}}^{\gamma'\alpha\beta'} \mathscr{D}_{\mathbf{l}}^{\gamma\gamma'} \mathscr{D}_{\mathbf{j}}^{\beta\beta'} + M_{\mathbf{ljk}}^{\gamma'\beta'\alpha} \mathscr{D}_{\mathbf{l}}^{\gamma\gamma'} \mathscr{D}_{\mathbf{j}}^{\beta\beta'}), \tag{5.22}$$

and when calculated agrees with the definition of (5.17). This derivation of $R_{\mathbf{k}}$ is of course little more than an argument, but it is not worth rigorizing it by this method for a proper derivation is given in the next section.

That the loss of energy should appear as a dynamical friction term was originally postulated by Heisenberg and the customary term 'turbulent viscosity' can be adopted for $R_{\mathbf{k}}$ (Hinze 1959) which will be seen to have precisely the same role as viscosity in (5.2), (5.3).

The essential point of this section lies in the need for using *two* functions $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ to describe the turbulent state. This has already been noted by Kraichnan (1959) in a paper which though differing in detail, has a very similar point of view to that of this section.

6. The general method of expansion

The basic assumption of this paper is that there exists a degree of randomness in the steady state of turbulence which permits the calculation of the turbulent viscosity. In this section a systematic expansion will be given for these two quantities, the expansion parameter being in effect the degree of randomness of the system. That of course is rather an inprecise statement, but a full discussion of what precisely one is expanding in terms of cannot be given without more experience of the expansion and its consequences, so it will be deferred to § 10. The time independent case is the simpler so it will be dealt with first. One needs then to find the solution of (2.22), in particular one may expect the solution to be simpler in an infinite volume than in a finite volume. In such a large volume the argument of Maxwell in kinetic theory will apply: that if in some region V_1 the probability distribution of the u is F_{V_1}, and in another adjacent region V_2 it is F_{V_2}, then taken together in the region V_1+V_2 the distribution function will be

$$F_{(V_1+V_2)} = F_{V_1} F_{V_2} \tag{6.1}$$

and hence

$$F_V = \mathcal{N} \exp\left\{\int_V d^3x\, Z([u])\right\}, \tag{6.2}$$

where Z is a functional of the u which contains no reference to the boundaries. For example, the case of a random external force which leads to (2.22) can be solved in the absence of the mixing term M to give

$$Z = \int d^3k\, u^{\alpha}_{\mathbf{k}} u^{\alpha}_{-\mathbf{k}} \gamma \mathbf{k}^2/h_{\mathbf{k}} \tag{6.3}$$

$$= \int d^3x \left\{\int d^3y \nabla^{\alpha} u^{\beta}(\mathbf{x})\, h^{-1}(\mathbf{x}-\mathbf{y})\, \nabla^{\alpha} u^{\beta}(\mathbf{y})\right\}. \tag{6.4}$$

In general, however, Z cannot be determined exactly and even if it could it would still leave a most intractable functional integral to be performed to obtain the various moments. Indeed the only functionals which are known to be integrable are polynomials multiplied by the exponentials of quadratic forms. Now the eigenfunctions of the kernel

$$\left\{\sum_{\mathbf{k},\alpha} \frac{\partial}{\partial u^{\alpha}_{\mathbf{k}}}\left(\sum_{\beta} \frac{\partial}{\partial \mathbf{u}^{\beta}_{\mathbf{k}}} \mathscr{D}^{\alpha\beta}_{\mathbf{k}} d_{\mathbf{k}} \Delta^{-1} + \omega_{\mathbf{k}} u^{\alpha}_{\mathbf{k}}\right) + \lambda\right\} \tag{6.5}$$

are the (complex) Hermite polynomials multiplied by the exponential

$$\exp\left(-\sum_{\mathbf{k}} u^{\alpha}_{\mathbf{k}} u^{\alpha}_{-\mathbf{k}} \Delta/q_{\mathbf{k}}\right).$$

Since the kernel is expected to play a central role in any expansion, this suggests expanding F as a series in the Hermite polynomials

$$H_{n_{\mathbf{k}} n_{\mathbf{k}}}(\mathbf{u}_{\mathbf{k}}, \mathbf{u}_{-\mathbf{k}})$$

defined by
$$H_{n_{\mathbf{k}}n_{\mathbf{k}}}(\mathbf{u_k},\mathbf{u_{-k}}) = [(n_{\mathbf{k}}+1)!]^{-\frac{1}{2}}[(n_{-\mathbf{k}}+1)!]^{-\frac{1}{2}},$$
$$\left(\frac{q_{\mathbf{k}}}{\Delta}\frac{\partial}{\partial u_{-\mathbf{k}}}-u_{\mathbf{k}}\right)^{n_{\mathbf{k}}}\left(\frac{q_{\mathbf{k}}}{\Delta}\frac{\partial}{\partial u_{\mathbf{k}}}-u_{-\mathbf{k}}\right)^{n_{-\mathbf{k}}}F_0, \tag{6.6}$$
where F_0 is given by (5.14).† Thus one writes
$$F = \sum_{\mathbf{n}} f_{[\mathbf{n}]}\prod_{\mathbf{k}} H_{n_{\mathbf{k}}n_{-\mathbf{k}}}F_0, \tag{6.7}$$
where $\mathbf{n}$ is a vector in the Hilbert space of all the Hermite polynomials
$$\mathbf{n} = (\dots n_{\mathbf{k}}, n_{-\mathbf{k}}\dots). \tag{6.8}$$
The eigenvalue associated with the label $\mathbf{n}$ is
$$\sum_{\mathbf{k}}\omega_{\mathbf{k}}n_{\mathbf{k}}. \tag{6.9}$$
It will be seen that the $n_{\mathbf{k}}$, $n_{-\mathbf{k}}$th polynomial is a tensor of rank $n_{\mathbf{k}}+n_{-\mathbf{k}}$. (The tensor indices are not written in explicitly.) Well-known relations exist between the polynomials such as
$$\frac{\partial}{\partial\mathbf{u_k}}H_{n_{\mathbf{k}}n_{-\mathbf{k}}} = \left(\frac{q_{\mathbf{k}}}{\Delta}\right)(n_{\mathbf{k}}+1)^{\frac{1}{2}}H_{n_{\mathbf{k}}+1,\,n_{-\mathbf{k}}}-n_{\mathbf{k}}^{\frac{1}{2}}H_{n_{\mathbf{k}}-1,\,n_{-\mathbf{k}}}, \tag{6.10}$$
$$\mathbf{u_k}H_{n_{\mathbf{k}}n_{-\mathbf{k}}} = (n_{\mathbf{k}}+1)^{\frac{1}{2}}H_{n_{\mathbf{k}}+1,\,n_{-\mathbf{k}}}+\frac{\Delta}{q_{\mathbf{k}}}n_{\mathbf{k}}^{\frac{1}{2}}H_{n_{\mathbf{k}}-1,\,n_{-\mathbf{k}}}, \tag{6.11}$$
but in practice only a few of the polynomials will be needed in an infinite system
$$\left.\begin{aligned} H_0 &= 1,\\ H_{1_{\mathbf{k}}} &= u_{\mathbf{k}}/\sqrt{2},\\ H_{1_{\mathbf{k}}1_{-\mathbf{k}}} &= (u_{\mathbf{k}}u_{-\mathbf{k}}-q_{\mathbf{k}}\Delta^{-1}).\end{aligned}\right\} \tag{6.12}$$
These polynomials are orthogonal to one another against the Gaussian F_0, but are not normalized to unity but to $(q_{\mathbf{k}}/\Delta)^{n_{\mathbf{k}}+n_{-\mathbf{k}}}$.

Consider the differential equation for F rearranged in the form
$$\left\{\sum_{\mathbf{k};\alpha}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\left(\sum_{\beta}\frac{\partial}{\partial u^{\beta}}\Delta^{-1}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}d_{\mathbf{k}}+\omega_{\mathbf{k}}u_{\mathbf{k}}^{\alpha}\right)-\sum_{\alpha\beta\gamma;\,\mathbf{kjl}}M_{-\mathbf{kjl}}^{\alpha\beta\gamma}u_{\mathbf{j}}^{\beta}u_{\mathbf{l}}^{\gamma}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\right.$$
$$\left.-\sum_{\mathbf{k},\alpha}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\left(\sum_{\beta}\frac{\partial}{\partial u_{\mathbf{k}}^{\beta}}\Delta^{-1}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}S_{\mathbf{k}}+R_{\mathbf{k}}u_{\mathbf{k}}^{\alpha}\right)\right\}F = 0. \tag{6.13}$$
Now, following the ideas of §§4 and 5, ascribe to S and R the (superficial) order M^2 and consider F expanded as a series in M. Then one has
$$F = F_0+F_1+F_2+\dots, \tag{6.14}$$
where, if the operator $\sum_{\mathbf{k},\alpha}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\left(\sum_{\beta}\frac{\partial}{\partial u_{\mathbf{k}}^{\beta}}\Delta^{-1}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}d_{\mathbf{k}}+\omega_{\mathbf{k}}u_{\mathbf{k}}^{\alpha}\right)$

is denoted by $\mathscr{K}$,
$$\mathscr{K}F_1 = \sum_{\mathbf{kjl};\,\alpha\beta\gamma}M_{\mathbf{kjl}}^{\alpha\beta\gamma}u_{\mathbf{j}}^{\beta}u_{\mathbf{l}}^{\gamma}\frac{\partial}{\partial u_{-\mathbf{k}}^{\alpha}}F_0 \tag{6.15}$$
$$\left.\begin{aligned}\mathscr{K}F_2 &= \sum_{\mathbf{kjl};\,\alpha\beta\gamma}M_{\mathbf{kjl}}^{\alpha\beta\gamma}u_{\mathbf{j}}u_{\mathbf{l}}\frac{\partial}{\partial u_{-\mathbf{k}}^{\alpha}}F_1+\sum_{\alpha}\frac{\partial}{\partial u_{-\mathbf{k}}^{\alpha}}\left(\sum_{\beta}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\Delta^{-1}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}S_{\mathbf{k}}+R_{\mathbf{k}}u_{\mathbf{k}}^{\alpha}\right)F_0\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}\right\} \tag{6.16}$$

† The suggestion that Hermite polynomials should be used for the expansion has also been made by Hopf (1962). I am grateful to Dr Kraichnan for this reference.

If the right-hand side of each equation is now rearranged in terms of the Hermite functions, one has then to solve equations of the type

$$\mathscr{H} F_l = \sum_{[\mathbf{n}]} C^l_{[\mathbf{n}]} \prod_{\mathbf{k}} H_{n_\mathbf{k} n_{-\mathbf{k}}} F_0, \tag{6.17}$$

which have the immediate solution

$$F_l = \sum_{[\mathbf{n}]} C^l_{[\mathbf{n}]} (\sum_{\mathbf{k}} n_\mathbf{k} \omega_\mathbf{k})^{-1} \prod_{\mathbf{k}} H_{n_\mathbf{k} n_{-\mathbf{k}}} F_0. \tag{6.18}$$

To complete the specification of the expansion one wishes to give correctly the mean value of $u^\alpha_\mathbf{k} u^\beta_{-\mathbf{k}}$ from F_0 alone, so that

$$\int (F_1 + F_2 + \dots) u^\alpha_\mathbf{k} u^\beta_{-\mathbf{k}} \Pi \, du_\mathbf{j} = 0. \tag{6.19}$$

Proceeding now to calculate F_1,

$$\mathscr{H} F_1 = \sum_{\mathbf{kjl};\alpha\beta\gamma} M^{\alpha\beta\gamma}_{\mathbf{kjl}} u^\beta_\mathbf{j} u^\gamma_\mathbf{l} u^\alpha_\mathbf{k} q^{-1}_\mathbf{k} \Delta F_0. \tag{6.20}$$

The right-hand side contains $H_{1_\mathbf{k}} H_{1_\mathbf{j}} H_{1_\mathbf{l}}$ so that

$$F_1 = \sum_{\alpha\beta\gamma;\mathbf{kjl}} M^{\alpha\beta\gamma}_{\mathbf{kjl}} u^\alpha_\mathbf{k} u^\beta_\mathbf{j} u^\gamma_\mathbf{l} \Delta q^{-1}_\mathbf{k} F_0 (\omega_\mathbf{k} + \omega_\mathbf{j} + \omega_\mathbf{l})^{-1}, \tag{6.21}$$

or in continuous variables

$$F_1 = \sum_{\alpha\beta\gamma} \frac{1}{(2\pi)^3} \int \frac{d^3k \, d^3j (2i)^{-1} (k^\beta \mathscr{D}^{\alpha\gamma}_\mathbf{k} + k^\gamma \mathscr{D}^{\alpha\beta}_\mathbf{k}) u^\alpha_\mathbf{k} u^\beta_\mathbf{j} u^\gamma_{-\mathbf{k}-\mathbf{j}}}{\omega_\mathbf{k} + \omega_\mathbf{j} + \omega_{-\mathbf{k}-\mathbf{j}}}. \tag{6.22}$$

The expression for F_2 is much more involved,

$$\begin{aligned}\mathscr{H} F_2 = \sum_{\mathbf{kjl};\alpha\beta\gamma} M^{\alpha\beta\gamma}_{\mathbf{kjl}} u^\beta_\mathbf{j} u^\gamma_\mathbf{l} \frac{\partial}{\partial u^\alpha_{-\mathbf{k}}} \sum_{\alpha'\beta'\gamma';\mathbf{k'j'l'}} (\omega_{\mathbf{k}'} + \omega_{\mathbf{j}'} + \omega_{\mathbf{l}'}) M^{\alpha'\beta'\gamma'}_{\mathbf{k'j'l'}} u^{\alpha'}_{\mathbf{k}'} u^{\beta'}_{\mathbf{j}'} u^{\gamma'}_{\mathbf{j}'} \Delta q^{-1}_{\mathbf{k}'} F_0 \\ + \sum_{\mathbf{k}} \left(\frac{S_\mathbf{k}}{q_\mathbf{k}} - R_\mathbf{k}\right) H_{1_\mathbf{k} 1_{-\mathbf{k}}} q^{-2}_\mathbf{k} \Delta.\end{aligned} \tag{6.23}$$

The resolution of the right-hand side into Hermite polynomials is straightforward but tedious and leads ultimately to the form

$$\begin{aligned}F_2 = \sum_{\substack{\text{all indices} \\ \text{all vectors}}} M^{\alpha\beta\gamma}_{\mathbf{kjl}} M^{\alpha'\beta'\gamma'}_{\mathbf{k'j'l'}} (\omega_{\mathbf{k}'} + \omega_{\mathbf{j}'} + \omega_{\mathbf{l}'})^{-1} q^{-1}_\mathbf{k} q^{-1}_{\mathbf{k}'} \Phi^{\alpha\beta\gamma\alpha'\beta'\gamma'}_{\mathbf{kjlk'j'l'}} \\ + \sum_{\alpha,\beta} (S_\mathbf{k} q^{-1}_\mathbf{k} - R_\mathbf{k}) H_{1_\mathbf{k} 1_{-\mathbf{k}}} q^{-2}_\mathbf{k} \Delta (\omega_\mathbf{k} + \omega_{-\mathbf{k}})^{-1},\end{aligned} \tag{6.24}$$

where, writing

$$H_\mathbf{k} \text{ for } H_{1_\mathbf{k}, 1_{-\mathbf{k}}}, \quad q^{\alpha\beta}_\mathbf{k} \text{ for } \mathscr{D}^{\alpha\beta}_\mathbf{k} q_\mathbf{k}, \quad \delta_{\mathbf{k}+\mathbf{j}} \text{ for } \delta_{\mathbf{k}+\mathbf{j}\,0}$$

and $\quad \omega_6 = \omega_\mathbf{k} + \omega_\mathbf{j} + \omega_\mathbf{l} + \omega_{\mathbf{k}'} + \omega_{\mathbf{j}'} + \omega_{\mathbf{l}'}$ ($\mathbf{k} \dots \mathbf{l}'$ all different),

$\omega_4 = \omega_\mathbf{a} + \omega_\mathbf{b} + \omega_\mathbf{c} + \omega_\mathbf{d}$ ($\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ all different, selected appropriately from $k \dots l'$),

$\omega_{4,2} = (\omega_\mathbf{a} + \omega_{-\mathbf{a}}) + (\omega_\mathbf{c} + \omega_\mathbf{d} + \omega_\mathbf{e} + \omega_\mathbf{f})$, ($\mathbf{a}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ all different as before),

$$\omega_2 = \omega_\mathbf{a} + \omega_{-\mathbf{a}}, \text{ etc.},$$

$$
\begin{aligned}
\Phi^{\alpha\beta\gamma\alpha'\beta'\gamma'}_{\mathbf{kjlk'j'l'}} = {} & 2u^{\beta}_{\mathbf{j}}u^{\gamma}_{\mathbf{l}}H^{\beta'\alpha}_{\mathbf{k}}u^{\gamma}_{\mathbf{l'}}u^{\alpha'}_{\mathbf{k'}}\delta_{\mathbf{k+j'}}\omega^{-1}_{4,\frac{1}{2}} \\
& + 4H^{\beta\gamma}_{\mathbf{j}}q^{\gamma\alpha'}_{\mathbf{l}}H^{\beta\alpha'}_{\mathbf{k}}\delta_{\mathbf{k+j'}}\delta_{\mathbf{j+l'}}\omega^{-1}_{2,\frac{1}{2}} + 4H^{\beta\gamma}_{\mathbf{j}}H^{\gamma\alpha'}_{\mathbf{l}}H^{\beta'\alpha}_{\mathbf{k}}\delta_{\mathbf{k+j'}}\delta_{\mathbf{j+l'}}\omega^{-1}_{2,\frac{1}{2},2} \\
& + 4q^{\beta\gamma'}_{\mathbf{j}}H^{\gamma\alpha'}_{\mathbf{l}}H^{\beta'\alpha}_{\mathbf{k}}\delta_{\mathbf{l-k}}\delta_{\mathbf{j+l'}}\omega^{-1}_{2,\frac{1}{2}} + u^{\beta}_{\mathbf{j}}u^{\gamma}_{\mathbf{l}}H^{\alpha\alpha'}_{\mathbf{k}}u^{\beta'}_{\mathbf{j}}u^{\gamma'}_{\mathbf{l'}}\delta_{\mathbf{k'-k}}\omega^{-1}_{4,\frac{1}{2}} \\
& + 2q^{\beta\beta'}_{\mathbf{j}}H^{\gamma\gamma'}_{\mathbf{l}}H^{\alpha\alpha'}_{\mathbf{k}}\delta_{\mathbf{k'-k}}\delta_{\mathbf{j'-j}}\omega^{-1}_{2,\frac{1}{2}} + 2H^{\beta\beta'}_{\mathbf{j}}q^{\gamma\gamma'}_{\mathbf{l}}H^{\alpha\alpha'}_{\mathbf{k}}\delta_{\mathbf{k'-k}}\delta_{\mathbf{j-j'}}\omega^{-1}_{2,\frac{1}{2}} \\
& + 2H^{\beta\beta'}_{\mathbf{j}}H^{\gamma\gamma'}_{\mathbf{l}}H^{\alpha\alpha'}_{\mathbf{k}}\delta_{\mathbf{k'-k}}\delta_{\mathbf{j'-j}}\omega^{-1}_{2,\frac{1}{2},2} + u^{\beta}_{\mathbf{j}}u^{\gamma}_{\mathbf{l}}u^{\alpha}_{\mathbf{k}}u^{\beta'}_{\mathbf{j'}}u^{\gamma'}_{\mathbf{l'}}u^{\alpha'}_{\mathbf{k'}}\omega_6^{-1} \\
& + 4u^{\gamma}_{\mathbf{l}}u^{\alpha}_{\mathbf{k}}u^{\gamma'}_{\mathbf{l'}}u^{\alpha'}_{\mathbf{k'}}q^{\beta\beta'}_{\mathbf{j}}\delta_{\mathbf{j-j'}}\omega_4^{-1} + 4u^{\gamma}_{\mathbf{l}}u^{\alpha}_{\mathbf{k}}u^{\gamma'}_{\mathbf{l'}}u^{\alpha'}_{\mathbf{k'}}H^{\beta\beta'}_{\mathbf{j}}\delta_{\mathbf{j-j'}}\omega^{-1}_{4,\frac{1}{2}} \\
& + 2u^{\gamma}_{\mathbf{l}}u^{\alpha}_{\mathbf{k}}u^{\beta'}_{\mathbf{j'}}u^{\gamma'}_{\mathbf{l'}}H^{\beta\alpha'}_{\mathbf{j}}\delta_{\mathbf{k'-j}}\omega^{-1}_{4,\frac{1}{2}} + 2u^{\gamma}_{\mathbf{l}}u^{\alpha}_{\mathbf{k}}u^{\beta'}_{\mathbf{j'}}u^{\gamma'}_{\mathbf{l'}}q^{\beta\alpha'}_{\mathbf{j}}\delta_{\mathbf{k'-j'}}\omega_4^{-1} \\
& + 2q^{\beta\beta'}_{\mathbf{j}}q^{\gamma\gamma'}_{\mathbf{l}}H^{\alpha\alpha'}_{\mathbf{k}}\delta_{\mathbf{k'-k}}\delta_{\mathbf{j'-j}}\omega_2^{-1} + 4q^{\beta\gamma'}_{\mathbf{j}}q^{\gamma\alpha'}_{\mathbf{l}}H^{\beta'\alpha}_{\mathbf{k}}\delta_{\mathbf{k+j'}}\delta_{\mathbf{j+l}}\omega_2^{-1}. \qquad (6.25)
\end{aligned}
$$

It is implied in the summations that all the u are different Fourier components, all products of the same component having been resolved into the polynomials. Now of all these terms only the last contributes to (6.19), and $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ are therefore chosen to make this vanish for each $\mathbf{k}$. By comparing the coefficients of $q_{\mathbf{k}}$ one sees that $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ are precisely those of the previous section. The remaining terms will give the values of

$$\langle u^{\alpha}_{\mathbf{k}}u^{\beta}_{\mathbf{j}}u^{\gamma}_{\mathbf{l}}u^{\delta}_{\mathbf{m}}\rangle \quad \text{and} \quad \langle u^{\alpha}_{\mathbf{k}}u^{\beta}_{\mathbf{j}}u^{\gamma}_{\mathbf{l}}u^{\delta}_{\mathbf{m}}u^{\epsilon}_{\mathbf{n}}u^{\nu}_{\mathbf{p}}\rangle$$

to this order of approximation. It is to be noted that the four-u correlation cannot be factorized into two $\langle uu\rangle$ correlations, and the six u cannot be factorized into $\Sigma\langle uu \times uuuu\rangle$.

Now in view of the complexity of F_2 it might be supposed that the higher approximations become unbearably complicated. This, however, is not the case when one passes to the infinitely large volume and a general procedure for writing down the nth term of the series is given in Appendix 4. At each stage new terms appear which give corrections to S and R in order that (6.19) be fulfilled. Some general results can be stated about this series: (i) The number of positive terms to any order equals the number of negative terms. (ii) The nth term of S can be written symbolically as

$$\int [M]^{2n}\,[q]^{n+1}\,[\Sigma\omega]^{-2n+1}\,(d^3k)^n$$

and the nth term in R as

$$\int [M]^{2n}\,[q]^{n}\,[\Sigma\omega]^{-2n+1}\,(d^3k)^n.$$

For example, the expressions (5.16) and (5.21), and typical terms of the next order which are (Appendix 4)

$$\Sigma\int\frac{M^{\alpha\beta\gamma}_{\mathbf{kjl}}M^{\gamma'\delta'\epsilon}_{\mathbf{jpm}}M^{\beta'\epsilon'\nu'}_{\mathbf{lpn}}M^{\delta\nu\alpha'}_{\mathbf{mnk}}q_{\mathbf{p}}q_{\mathbf{m}}q_{\mathbf{n}}\mathscr{D}^{\beta\beta'}_{\mathbf{j}}\mathscr{D}^{\delta\delta'}_{\mathbf{m}}\mathscr{D}^{\gamma\gamma'}_{\mathbf{j}}\mathscr{D}^{\epsilon\epsilon'}_{\mathbf{p}}\mathscr{D}^{\nu\nu'}_{\mathbf{n}}}{(\omega_{\mathbf{k}}+\omega_{\mathbf{n}}+\omega_{\mathbf{m}})(\omega_{\mathbf{k}}+\omega_{\mathbf{m}}+\omega_{\mathbf{l}}+\omega_{\mathbf{p}})(\omega_{\mathbf{k}}+\omega_{\mathbf{l}}+\omega_{\mathbf{j}})}\,d^3l\,d^3p\,d^3m\,d^3n$$

and

$$\Sigma\int\frac{M^{\alpha\beta\gamma}_{\mathbf{kjl}}M^{\gamma'\delta'\epsilon}_{\mathbf{jpm}}M^{\beta'\epsilon'\nu'}_{\mathbf{lpn}}M^{\delta\nu\alpha'}_{\mathbf{mnk'}}q_{\mathbf{p}}q_{\mathbf{n}}\mathscr{D}^{\beta\beta'}_{\mathbf{l}}\mathscr{D}^{\delta\delta'}_{\mathbf{m}}\mathscr{D}^{\gamma\gamma'}_{\mathbf{j}}\mathscr{D}^{\epsilon\epsilon'}_{\mathbf{p}}\mathscr{D}^{\nu\nu'}_{\mathbf{n}}}{(\omega_{\mathbf{k}}+\omega_{\mathbf{n}}+\omega_{\mathbf{m}})(\omega_{\mathbf{k}}+\omega_{\mathbf{m}}+\omega_{\mathbf{l}}+\omega_{\mathbf{p}})(\omega_{\mathbf{k}}+\omega_{\mathbf{l}}+\omega_{\mathbf{j}})}\,d^3l\,d^3j\,d^3p\,d^3m\,d^3n$$

for S and R, respectively. In the symbolic notation these are

$$\int [M]^4\,[q]^3\,[\Sigma\omega]^{-3}\,[d^3k]^2 \quad \text{and} \quad \int [M]^4\,[q]^2\,[\Sigma\omega]^{-3}\,[d^3k]^2,$$

the M containing δ-functions which remove three integrations.

This completes the discussion of the steady distribution, and in the next section the time-dependent case is discussed.

7. The general expansion in the generalized phase space

To resolve the time-dependent case the same method as that of § 6 will be employed. There is no point now in averaging out the random input force beforehand, so the expansion will be made in $\mathscr{F}$ as well as M, they being considered of the same order in as much as both are approximately random as was implicit in § 6. At this stage it is worth noting that there is no need for $\mathscr{F}$ to have the Gaussian distribution of (2.17) and independent values can be given for say $\langle \mathscr{F}\mathscr{F}\mathscr{F}\mathscr{F}\rangle$. These will however affect only the corrections to $\mathscr{S}$ and $\mathscr{R}$ being corrections to the basic assumption of randomness. The equation

$$\int dk_0 \sum_{\mathbf{k},\alpha} \frac{\partial}{\partial u_k^\alpha}\{(ik_0+\nu\mathbf{k}^2)\,u_k^\alpha - \sum_{j,l;\,\alpha\beta\gamma} M_{-kjl}^{\alpha\beta\gamma} u_j^\beta u_l^\gamma - \Sigma \mathscr{D}_{\mathbf{k}}^{\alpha\beta}\mathscr{F}_k^\beta\}\,P = 0 \tag{3.5}$$

will be rearranged as

$$\int dk_0 \sum_{\mathbf{k},\alpha}\left\{\frac{\partial}{\partial u_k^\alpha}\left(\sum_\beta \frac{\partial}{\partial u_k^\beta}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}(\Delta dk_0)^{-1}D_k + \Omega_k u_k^\alpha\right) - \sum_{\beta\gamma;\,jl} M_{-kjl}^{\alpha\beta\gamma} u_j^\beta u_l^\gamma \frac{\partial}{\partial u_k^\alpha}\right.$$
$$\left. - \sum_\beta \mathscr{D}_{\mathbf{k}}^{\alpha\beta}\mathscr{F}_k^\beta \frac{\partial}{\partial u_k^\alpha} - \frac{\partial}{\partial u_k^\alpha}\left(\sum_\beta \frac{\partial}{\partial u_{-k}^\beta}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}(\Delta dk_0)^{-1}D_k + \mathscr{R}_k u_k^\alpha\right)\right\} P = 0, \tag{7.1}$$

where
$$\Omega_k = \mathscr{R}_k + \nu\mathbf{k}^2 + ik_0. \tag{7.2}$$

The quantities $\mathscr{S}_k$, $\mathscr{R}_k$, Ω_k, D_k will appear analogues of $S_\mathbf{k}$, $R_\mathbf{k}$, $\omega_\mathbf{k}$, $d_\mathbf{k}$. Expanding as before

$$P = P_0 + P_1 + P_2 + \ldots, \tag{7.3}$$

it will now be required that

$$\int u_k^\alpha u_{-k}^\beta \langle P_1 + P_2 + \ldots\rangle \,\Pi\, du = 0, \tag{7.4}$$

where the averaging is over the distribution of the force $\mathscr{F}$ which will still be given by (2.17) and (2.20). The analysis goes through exactly as before, for example,

$$P_1 = \sum_{kjl;\,\alpha\beta\gamma} \frac{M_{kjl}^{\alpha\beta\gamma} u_j^\beta u_l^\gamma u_k^\alpha \mathscr{Q}_k^{-1}}{\Omega_k + \Omega_j + \Omega_l} + \sum_{k;\,\alpha,\beta} \frac{\mathscr{D}_{\mathbf{k}}^{\alpha\beta}\mathscr{F}_k^\alpha u_{-k}^\beta \mathscr{Q}_k^{-1}}{\Omega_k} \tag{7.5}$$

so that only the results will be quoted. Since it is still true that the mean rate of input of energy is given by $\langle \mathscr{F}_k \mathscr{F}_{-k}\rangle = h_\mathbf{k}$

one may define
$$\mathscr{S}_k = D_k - h_\mathbf{k}/\Omega_k. \tag{7.6}$$

Then if
$$\Delta dk_0 \langle u_k^\alpha u_{-k}^\beta\rangle = \mathscr{Q}_k \mathscr{D}_{\mathbf{k}}^{\alpha\beta} \tag{7.7}$$

$$\mathscr{S}_k = \int \frac{L_{\mathbf{kjl}}\mathscr{Q}_j \mathscr{Q}_l d^3j\, dj_0\, d^3l\, dl_0}{\Omega_k + \Omega_j + \Omega_l}\,\delta(k_0 + j_0 + l_0), \tag{7.8}$$

$$\mathscr{R}_k = \int \frac{L_{\mathbf{kjl}}\mathscr{Q}_j d^3j\, dj_0\, d^3l\, dl_0\, \delta(k_0 + j_0 + l_0)}{\Omega_k + \Omega_j + \Omega_l} \tag{7.9}$$

and
$$\mathscr{Q}_k = D_k/\Omega_k. \tag{7.10}$$

There is one great simplification to be noted in these equations. Since

$$k_0 + j_0 + l_0 = 0 \tag{7.11}$$

it follows that
$$\Omega_k + \Omega_j + \Omega_l = \mathscr{R}_k + \mathscr{R}_j + \mathscr{R}_l + \nu\mathbf{k}^2 + \nu\mathbf{j}^2 + \nu\mathbf{l}^2. \tag{7.12}$$

If one introduces Q_k by the definition

$$\mathscr{Q}_k^{\alpha\alpha'} = Q_k q_{\mathbf{k}} \mathscr{D}_{\mathbf{k}}^{\alpha\alpha'}, \tag{7.13}$$

then from the definitions of $\mathscr{Q}$ and q it follows that

$$\int Q_k \, dk_0 = 1. \tag{7.14}$$

If one now tries $\mathscr{R}_k = R_{\mathbf{k}}$ as a solution of (7.9) it does indeed satisfy it. But $R_{\mathbf{k}}$ can be taken as known since the equations for it are independent of $\mathscr{S}_{\mathbf{k}}$ and $Q_{\mathbf{k}}$. It follows that one can now write a closed equation for Q_k

$$(ik_0 + \nu\mathbf{k}^2) Q_k + h_{\mathbf{k}}(\Omega_k q_{\mathbf{k}})^{-1} + \int \frac{L_{\mathbf{kjl}} q_{\mathbf{j}} q_{\mathbf{l}} Q_j Q_{-k-j} \, dj_0 \, d^3j \, d^3l}{q_{\mathbf{k}}(\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}})} = 0 \tag{7.15}$$

in which $q_{\mathbf{k}}$, $\omega_{\mathbf{k}}$, Ω_k, $L_{\mathbf{kjl}}$ all have the same meaning as before. This can usefully be written in time-dependent form by Fourier transformation

$$\left(\frac{\partial}{\partial t} + \omega_{\mathbf{k}}\right) Q_{\mathbf{k}}(t) = \frac{h_{\mathbf{k}}}{q_{\mathbf{k}}} e^{-\omega_{\mathbf{k}} t} + \int d^3j \, \zeta(\mathbf{j}, \mathbf{k}) \, Q_{\mathbf{j}}(t) \, Q_{-\mathbf{k}-\mathbf{j}}(t), \tag{7.16}$$

where
$$Q_{\mathbf{k}}(t) = \frac{1}{2\pi} \int Q_k e^{ik_0 t} \, dk_0 \tag{7.17}$$

and ζ is the kernel of the integral in (7.15).

The extension to higher approximation goes through exactly as before and in the time-dependent form the remark (i) still holds, as of course does remark (ii) when $d^3k\,dk_0$ replaces d^3k.

An expansion has now been obtained in the time-independent and time-dependent cases for the distribution function. By analogy with other branches of theoretical physics it may be termed the generalized random-phase approximation. To understand its implications one needs to solve the equations in as many cases as one can, and it will turn out that it is possible to make considerable progress in spite of the complexity of the equations.

It is important to emphasize that the expansion developed here is quite different from those obtained by truncating the infinite set of equations got by taking moments of the Navier–Stokes equations, which have a structure similar to the equations developed in quantum-field theory. These approaches in effect try to make $R_{\mathbf{k}}$, $\mathscr{R}_k$ do the work of both $(S_{\mathbf{k}}, \mathscr{S}_k,)$, $(R_{\mathbf{k}}, \mathscr{R}_k)$ and the form of the solution suggested can be got from the forms above by taking, in say the time-independent case,

$$q_{\mathbf{k}} = (h_{\mathbf{k}} + S_{\mathbf{k}})/(\nu\mathbf{k}^2 + R_{\mathbf{k}})$$

and dividing the numerator into the denominator

$$\begin{aligned} q_{\mathbf{k}} &= h_{\mathbf{k}}(\nu\mathbf{k}^2 + R_{\mathbf{k}})^{-1}(1 + S_{\mathbf{k}}/h_{\mathbf{k}})^{-1} \\ &\simeq h_{\mathbf{k}}(\nu\mathbf{k}^2 + R_{\mathbf{k}} - \nu\mathbf{k}^2 S_{\mathbf{k}}/h_{\mathbf{k}})^{-1}. \end{aligned}$$

Now expand the integral for $R_{\mathbf{k}}$

$$R_{\mathbf{k}} = \int L_{\mathbf{ljk}} q_{\mathbf{j}} (\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}})^{-1} d^3j\, d^3l.$$

$$\cong \int L_{\mathbf{ljk}} q_j (\omega_{\mathbf{j}} + \omega_{\mathbf{l}})^{-1} d^3j\, d^3l - \omega_{\mathbf{k}} \int L_{\mathbf{ljk}} q_j (\omega_{\mathbf{j}} + \omega_{\mathbf{l}})^{-2} d^3j\, d^3l.$$

If one now writes $$q_{\mathbf{k}} = h_{\mathbf{k}} (\nu \mathbf{k}^2 + \Sigma_{\mathbf{k}})^{-1}$$

and puts further $\omega_{\mathbf{k}} \cong \nu\mathbf{k}^2 + \Sigma_{\mathbf{k}}$, then

$$q_{\mathbf{k}} \cong h_{\mathbf{k}} \left(\nu\mathbf{k}^2 + \frac{1}{2} \int L_{\mathbf{ljk}} q_{\mathbf{j}} q_{\mathbf{l}} d^3j\, d^3l \right)^{-1},$$

which is a form often studied, and typical of the kind of expression obtained by manipulating the Navier–Stokes equations directly. There is no underlying physical plausibility for this form, however, and though rather complicated mathematical manoeuvres have been performed above, they follow as closely as possible the intuitive models of earlier sections. The relation of the equations derived here with the work of Kolmogoroff and Kraichnan will be discussed at the end of the next section.

8. Properties and solutions of the equations

Before attempting to solve the equations derived in §§ 6 and 7 one must verify that the expansion of P is in accord with the original Navier–Stokes equations from which the whole analysis stemmed. To see that this is the case multiply the original Navier–Stokes equation (2.8) taken at the time t, by $U_{\mathbf{k}}$ at the time t', and average. This gives

$$\frac{\partial}{\partial t} \langle U^{\alpha}_{-\mathbf{k}}(t)\, U^{\alpha'}_{\mathbf{k}}(t') \rangle + \nu\mathbf{k}^2 \langle U^{\alpha}_{-\mathbf{k}}(t)\, U^{\alpha'}_{\mathbf{k}}(t') \rangle$$
$$- \sum_{\beta,\gamma;\,\mathbf{j},\mathbf{l}} M^{\alpha\beta\gamma}_{\mathbf{kjl}} \langle U^{\beta}_{\mathbf{j}}(t)\, U^{\gamma}_{\mathbf{l}}(t)\, U^{\alpha'}_{\mathbf{k}}(t') \rangle + \sum_{\beta} \mathscr{D}^{\alpha\beta}_{\mathbf{k}} \langle \mathscr{F}^{\beta}_{-\mathbf{k}}(t)\, U^{\alpha'}_{\mathbf{k}}(t') \rangle = 0, \quad (8.1)$$

or in the four-dimensional Fourier transform

$$\mathscr{D}^{\alpha\alpha'}_{\mathbf{k}} (ik_0 - \nu\mathbf{k}^2)\, \mathscr{Q}_k - \sum_{\beta} \mathscr{D}^{\alpha\beta}_{\mathbf{k}} \langle \mathscr{F}^{\beta}_k U^{\alpha'}_k \rangle + \sum_{\beta,\gamma;\,\mathbf{j},\mathbf{l}} M^{\alpha\beta\gamma}_{kjl} \langle U^{\beta}_j U^{\gamma}_l U^{\alpha'}_k \rangle = 0. \quad (8.2)$$

Using the expansion for P, the two terms on the right are evaluated from P_1 (7.5), and give $h_{\mathbf{k}}/\Omega_k$ and $\mathscr{S}_k - R_{\mathbf{k}} \mathscr{Q}_k$ respectively, so that the original equation implies that

$$(ik_0 - \nu\mathbf{k}^2)\, \mathscr{Q}_k + \frac{h_{\mathbf{k}}}{\Omega_k} + (\mathscr{S}_k - R_k \mathscr{Q}_k) = 0 \quad (8.3)$$

which is indeed (7.15).

A similar result applies to the time-independent case and there represents a discussion of the flow of energy. The total energy is

$$\mathscr{E} = \frac{1}{2} \int U^{\alpha}_{\mathbf{k}} U^{\alpha}_{-\mathbf{k}} d^3k; \quad (8.4)$$

hence, at any time, the ratio of change of energy is given by

$$\frac{\partial \mathscr{E}}{\partial t} = \int \nu \mathbf{k}^2 q_{\mathbf{k}}\, d^3k + \int h_{\mathbf{k}} d^3k + \int d^3k\, d^3j\, d^3l \left\langle \sum_{\alpha\beta\gamma} M^{\alpha\beta\gamma}_{\mathbf{kjl}} u^\beta_{\mathbf{j}} u^\gamma_{\mathbf{l}} u^\alpha_{\mathbf{k}} \right\rangle. \tag{8.5}$$

Now the term in M vanishes by symmetry, but it can also be written from F_1 and gives

$$\int d^3k(S_k - R_k q_k). \tag{8.6}$$

But the definitions of S and R ensure that this expression vanishes for

$$\int d^3k\, S_k = \int \frac{L_{kjl} q_j q_l}{\omega_k + \omega_j + \omega_l}\, d^3k\, d^3j\, d^3l, \tag{8.7}$$

$$\int d^3k\, R_k q_k = \int \frac{L_{ljk} q_j q_k}{\omega_k + \omega_j + \omega_l}\, d^3k\, d^3j\, d^3l. \tag{8.8}$$

It follows that the total external input and output balance and also the total internal input and output, as indeed must be the case since no work is done by the inertial terms. Whereas S and Rq being rates at which energy is absorbed or emitted are familiar concepts, the conservation properties are also true of $\mathscr{S}$ and $R\mathscr{Q}$ which refer to *action*.

Turning now to the solution of the equations, the simplest case is clearly that of §6, so one may ask whether there are *any* conditions, however remote from physical attainability, under which an exact solution can be obtained. It is a property of M that, bearing in mind that $\operatorname{div} \mathbf{u} = 0$, one may rewrite

$$\sum_{\alpha\beta\gamma;\,\mathbf{kjl}} \frac{M^{\alpha\beta\gamma}_{\mathbf{kjl}} u^\beta_{\mathbf{j}} u^\gamma_{\mathbf{l}} u^\alpha_{\mathbf{k}} q^{-1}_{\mathbf{k}}}{\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}}}, \tag{8.9}$$

in the form

$$\tfrac{1}{2} \sum_{\alpha\beta\gamma;\,\mathbf{kjl}} M^{\alpha\beta\gamma}_{\mathbf{kjl}} u^\beta_{\mathbf{j}} u^\gamma_{\mathbf{l}} u^\alpha_{\mathbf{k}} (q^{-1}_{\mathbf{k}} - q^{-1}_{\mathbf{j}}). \tag{8.10}$$

It follows that if $q_{\mathbf{k}}$ is a constant, not only F_1, but all higher terms vanish identically, and it is clear from their definitions that in this case

$$S_{\mathbf{k}} = R_{\mathbf{k}} q_{\mathbf{k}}. \tag{8.11}$$

From this and equation (5.4) one has

$$h_{\mathbf{k}} = \nu \mathbf{k}^2 q_{\mathbf{k}}, \tag{8.12}$$

so that constant $q_{\mathbf{k}}$ will be a solution if $h_{\mathbf{k}}$ is taken to be

$$h_{\mathbf{k}} = h(|\mathbf{k}|\, k_1^{-1})^2, \tag{8.13}$$

$$q = h\nu^{-1} k_1^{-2}. \tag{8.14}$$

If h and ν both tend to zero, their ratio is arbitrary and one can write $q = (2\kappa T)^{-1}$, since this is the case of thermal equilibrium, but in general q is well defined. For constant q the actual integrals for S and R are divergent which is scarcely surprising with an input rising like $\mathbf{k}^2$. But there still stems from this analysis the useful comment made earlier that in the corrections to S and R the number of positive terms to any order equals the number of negative terms, for it is only by this means that every order vanishes when q is constant (both q, ω are of course positive definite).

To consider more realistic cases one can simplify the equations by assuming that the input is more concentrated near small k so that the viscosity can be ignored in the first approximation. This is equivalent to the statement that the Reynolds number of the turbulence may be considered infinite. Of course one cannot balance input and output in the absence of viscosity, but this point can be resolved as will be shown. The simplest input is a power and it is possible to solve this case in the limit of infinite Reynolds number. So consider

$$h_{\mathbf{k}} = h(|\mathbf{k}|\,k_1^{-1})^{-\alpha}, \tag{8.15}$$

in which case the equation for $R_{\mathbf{k}}$ contains no dimensional parameters, and therefore $q_{\mathbf{k}}$, $R_{\mathbf{k}}$ must be powers, and therefore also $S_{\mathbf{k}}$. Define q, R by the equations

$$q_{\mathbf{k}} = q|\mathbf{k}|^{-m}, \tag{8.16}$$

$$R_{\mathbf{k}} = R|\mathbf{k}|^{n}. \tag{8.17}$$

Then from (5.21)

$$R|\mathbf{k}|^n = qR^{-1}\int d^3l\,d^3j\,|\mathbf{j}|^{-m}\,L_{\mathbf{kjl}}(|\mathbf{k}|^n+|\mathbf{j}|^n+|\mathbf{l}|^n)^{-1}. \tag{8.18}$$

Writing $\qquad |\mathbf{j}| = |\mathbf{j}'|\,|\mathbf{k}|, \quad |\mathbf{l}| = |\mathbf{l}'|\,|\mathbf{k}| = |\mathbf{l}+\mathbf{j}'|\,|\mathbf{k}|$

and using the explicit form of $L(\sim |\mathbf{j}|^2)$, one finds

$$R^2|\mathbf{k}|^n = q|\mathbf{k}|^{5-m-n}A^2, \tag{8.19}$$

where A^2 is a numerical constant, independent of $\mathbf{k}$. Therefore

$$R = A\sqrt{q}, \tag{8.20}$$

and

$$5 = 2n+m. \tag{8.21}$$

In the same way from (5.16),

$$S_{\mathbf{k}} = q^2R^{-1}|\mathbf{k}|^{5-2m-n}. \tag{8.22}$$

From the definition of $q_{\mathbf{k}}$ one now has

$$q|\mathbf{k}|^{-m} = \{h(|\mathbf{k}|\,k_1^{-1})^{-\alpha}+q^2R^{-1}|\mathbf{k}|^{5-2m-n}\}\,R^{-1}|\mathbf{k}|^{-n}, \tag{8.23}$$

and it follows that

$$5-2m-n = -\alpha, \tag{8.24}$$

$$-m+n = -\alpha, \tag{8.25}$$

i.e.

$$n = \tfrac{1}{3}(5-\alpha), \tag{8.26}$$

$$m = \tfrac{1}{3}(5+2\alpha) \tag{8.27}$$

and that

$$\sqrt{A}\,q^{\frac{3}{2}} = hk_1^{\alpha}+q^{\frac{3}{2}}BA^{-\frac{1}{2}}, \tag{8.28}$$

i.e.

$$q = h^{\frac{2}{3}}(A^{\frac{1}{2}}-BA^{-\frac{1}{2}})^{-\frac{2}{3}}\,k_1^{\frac{2}{3}\alpha}. \tag{8.29}$$

This result is not restricted to the approximations to $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ of equations (5.16) and (5.19) but is true to all orders, as is seen by considering the symbolic expressions for the higher terms quoted in § 6. The effect of the higher terms is to alter the coefficients which occur. It follows that there exist constants κ_α, ρ_α, σ_α such that the solution is given exactly by

$$\left.\begin{aligned} q_{\mathbf{k}} &= \kappa_\alpha h^{\frac{2}{3}} k_1^{-\frac{2}{3}\alpha} |\mathbf{k}|^{\frac{1}{3}(5+2\alpha)}, \\ R_{\mathbf{k}} &= \rho_\alpha h^{\frac{1}{3}} k_1^{\frac{1}{3}\alpha} |\mathbf{k}|^{\frac{1}{3}(5-\alpha)}, \\ S_{\mathbf{k}} &= \sigma_\alpha h k_1^{-\alpha} |\mathbf{k}|^{\alpha}. \end{aligned}\right\} \tag{8.30}$$

(In the literature it is customary to use the distribution per unit $|\mathbf{k}|$, i.e.

$$E(|\mathbf{k}|)\, dk = q_{\mathbf{k}} d^3k$$

so the above result gives $\qquad E(|\mathbf{k}|) \propto h^{\frac{2}{3}} k^{-\frac{1}{3}(2\alpha-1)}$.)

A particular case is that of white noise $\alpha = 0$ (white that is with respect to wavelength; the definition of $h_{\mathbf{k}}$ has already assumed whiteness with respect to frequency), for which

$$\left.\begin{aligned} q_{\mathbf{k}} &= \kappa h^{\frac{2}{3}} |\mathbf{k}|^{-\frac{5}{3}}, \\ R_{\mathbf{k}} &= \rho h^{\frac{1}{3}} |\mathbf{k}|^{\frac{5}{3}}, \\ S_{\mathbf{k}} &= \sigma h. \end{aligned}\right\} \tag{8.31}$$

The validity of these solutions depends on the integrals A and B, and the integrals appearing in higher approximations, all converging. It is $R_{\mathbf{k}}$ which is the critical function for if it exists it is easily confirmed that $S_{\mathbf{k}}$ and all the higher functions also exist. For small $|\mathbf{j}|$

$$L \sim |\mathbf{k}|^2,$$

so that the $R_{\mathbf{k}}$ integral goes like

$$\int dj\, |\mathbf{j}|^2\, |\mathbf{j}|^{\frac{1}{3}(2\alpha-5)}, \tag{8.32}$$

which implies that $\alpha < 2$. For large j, L contains terms like $\mathbf{k}.\mathbf{j}$ and $\mathbf{k}^2$ but allowing for the angular integration both give the expression

$$\int |\mathbf{j}|^2 dj\, |\mathbf{j}|^{-\frac{1}{3}(5-\alpha)-\frac{1}{3}(5+2\alpha)}, \tag{8.33}$$

implying that $\alpha > -1$. The solution then is valid for

$$2 > \alpha > -1. \tag{8.34}$$

For $\alpha < -1$, a solution cannot be obtained without invoking the viscosity which then affects the solution for all $|\mathbf{k}|$, as in the first example of this discussion. For $\alpha > 2$ the precise nature of the input at small $|\mathbf{k}|$ can be expected to affect the entire solution, and the input itself will of course be modified at small $|\mathbf{k}|$ in order that the total input be finite. This case will be discussed further shortly.

One has then a picture of energy entering the $\mathbf{k}$th component of the system at a rate $\Delta h(|\mathbf{k}|/k_1)^{-\alpha}$, gradually being transported to higher and higher $|\mathbf{k}|$ till finally the viscosity can no longer be ignored, with the result that $q_{\mathbf{k}}$ falls away very much faster. Finally, a region is reached in which $R_{\mathbf{k}}$ is negligible and only external input and output matter giving

$$h(|\mathbf{k}|/k_1)^{-\alpha}/\nu|\mathbf{k}|^2. \tag{8.35}$$

The apparent paradox that one may investigate the small $|\mathbf{k}|$ region by dropping the viscosity, even though the latter is required for overall energy balance, is resolved by observing that in the absence of viscosity the integrals over $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ do not converge. Large $|\mathbf{k}|$ values act as an infinite capacity well for energy, and the formal identity of the integrals over $|\mathbf{k}|$ of $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ is not meaningful. The inclusion of the viscosity restores the precision of the identity.

There is in the model of turbulence with source $h(|\mathbf{k}|/k_1)^{-\alpha}$, a localization of influence of one $|\mathbf{k}|$ value upon another. Thus if one takes a region δ of $\mathbf{k}$ space near to $\mathbf{k}_2$ say and asks how much this region contributes to $S_{\mathbf{k}}$, the answer is an amount which tends to zero as $\mathbf{k}_2+\mathbf{k}_1$ increases. To be precise

$$dS_{k_1} \sim (k_1+k_2)^2 q_{\mathbf{k}_1+\mathbf{k}_2} q_{\mathbf{k}_2} R^{-1}_{\mathbf{k}_1+\mathbf{k}_2} \delta$$

which tends to zero as $\mathbf{k}_1+\mathbf{k}_2$ tends to infinity. A similar statement holds for $R_{\mathbf{k}}$; both remarks are essentially contained in the statements that the integrals over all $|\mathbf{k}|$ space for $S_{\mathbf{k}}$ and $R_{\mathbf{k}}$ converge. Consequently one can regard the situation as a cascade of energy from small $|\mathbf{k}|$ to larger $|\mathbf{k}|$, supplemented by an external input which, per component, decreases as $|\mathbf{k}|$ increases. Finally, for large $\mathbf{k}$ viscosity comes in and kills the flow of energy. Of course it is clear from the definitions of $R_{\mathbf{k}}$ and $S_{\mathbf{k}}$ that energy flows in and out of all components to all components, of larger and smaller $\mathbf{k}$; but on an average it flows from small to large $\mathbf{k}$, the imbalance being taken up by the input.

The above model is still not satisfactory, however, since though the input per component decreases, the input per unit wave-number does not, the latter incorporating a weight factor $|\mathbf{k}|^2$ from $|\mathbf{k}|^2dk$, which overcomes the $|\mathbf{k}|^{-\alpha}$. A physically realistic model will decrease much faster than a power and one should expect $h_{\mathbf{k}}$ to be zero for $|\mathbf{k}| > K_1$ say. (This remark need not apply in magneto-hydrodynamic turbulence where white noise from electromagnetic sources is quite feasible and the previous model therefore significant.) Consider then such an input,

$$h_{\mathbf{k}} = 0, \quad |\mathbf{k}| > K_1. \tag{8.36}$$

Integrate the equation

$$h_{\mathbf{k}} = R_{\mathbf{k}} q_{\mathbf{k}} - S_{\mathbf{k}}$$

up to a value K, where $K > K_1$. If one introduces

$$\mathscr{H} = \int_0^\infty h_{\mathbf{k}} d^3k \tag{8.37}$$

$$= \int_0^{K_1} h_{\mathbf{k}} d^3k, \tag{8.38}$$

then

$$\mathscr{H} = \int d^3j\, d^3l \int_{|\mathbf{k}|\leqslant K} \left\{\frac{L_{\mathbf{ljk}} q_{\mathbf{j}} q_{\mathbf{k}} - L_{\mathbf{kjl}} q_{\mathbf{j}} q_{\mathbf{l}}}{R_{\mathbf{k}}+R_{\mathbf{j}}+R_{\mathbf{l}}}\right\} d^3k. \tag{8.39}$$

Since L contains a term $\delta(\mathbf{k}+\mathbf{j}+\mathbf{l})$ one may write

$$L_{\mathbf{ljk}} = \Lambda_{\mathbf{kj}}\, \delta(\mathbf{k}+\mathbf{j}+\mathbf{l}), \tag{8.40}$$

and so, by writing $\mathbf{k}+\mathbf{j}$ for $-\mathbf{k}$ in the second term obtain

$$\mathscr{H} = \int d^3j \left\{\int_{|\mathbf{k}|<K} \frac{\Lambda_{\mathbf{kj}} q_{\mathbf{k}} q_{\mathbf{j}}\, d^3k}{R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}}} - \int_{|\mathbf{k}+\mathbf{j}|<K} \frac{\Lambda_{\mathbf{kj}} q_{\mathbf{k}} q_{\mathbf{j}}\, d^3k}{R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}}}\right\}$$

$$= \int d^3j \int_\Sigma \Lambda_{\mathbf{kj}} q_{\mathbf{k}} q_{\mathbf{j}} (R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}})^{-1}\, d^3k, \tag{8.41}$$

where Σ is the region between the two spheres $|\mathbf{k}| = K$, $|\mathbf{k}+\mathbf{j}| = K$. Now suppose tentatively that the solution to this problem is again a power law, with

$$q_{\mathbf{k}} = q|\mathbf{k}|^{-m}, \quad R_{\mathbf{k}} = R|\mathbf{k}|^{-n}. \tag{8.42}$$

Then writing $|\mathbf{k}| = K|\mathbf{k}'|$ and $|\mathbf{j}| = K|\mathbf{j}'|$ one sees that

$$\mathscr{H} = K^{8-2m-n}\chi q^2 R^{-1}, \tag{8.43}$$

where χ is the value of the integral, a dimensionless constant. But $\mathscr{H}$ is a constant, independent of K, so one has

$$8 = 2m+n, \tag{8.44}$$

which when combined with the equation for $R_{\mathbf{k}}$, i.e. with (8.21), gives

$$\left.\begin{aligned} m &= \tfrac{11}{3}, \\ n &= \tfrac{2}{3}. \end{aligned}\right\} \tag{8.45}$$

This means that

$$q_{\mathbf{k}} = |\mathbf{k}|^{-\frac{11}{3}}\mathscr{H}^{\frac{2}{3}}(\chi/A)^{-\frac{2}{3}}, \tag{8.46}$$

or in terms of the distribution per unit wave-number

$$\begin{aligned} E(|\mathbf{k}|)\,dk &= 4\pi q_{\mathbf{k}}|\mathbf{k}|^2\,dk \\ &= 4\pi(\chi/A)^{\frac{2}{3}}\mathscr{H}^{\frac{2}{3}}|\mathbf{k}|^{-\frac{5}{3}}\,dk, \end{aligned} \tag{8.47}$$

the Kolmogoroff spectrum (see, for example, Batchelor 1959). Unfortunately the problem cannot be resolved so simply for the dimensional arguments only apply if all the integrals converge, and they do not. The relation between $q_{\mathbf{k}}$ and $R_{\mathbf{k}}$ is still (5.19) and still only converges if $\int |\mathbf{j}|^2 q_{\mathbf{j}}\,dj$ exists for small $|\mathbf{j}|$. This is clearly not the case for a $|\mathbf{j}|^{-\frac{11}{3}}$ dependence. Now of course at small $|\mathbf{j}|$ one will expect $q_{\mathbf{j}}$ to depart from $|\mathbf{j}|^{-\frac{11}{3}}$ and directly show the structure of the input. But the fact that one cannot still use $|\mathbf{j}|^{-\frac{11}{3}}$ near $|\mathbf{j}| \sim 0$ in the integrals implies that the effect of the deviation will make itself felt everywhere and hence the structure of the input will be noticed everywhere. Unlike the power input model, this case does not form a simple cascade. To investigate what happens more fully, examine the equation for $R_{\mathbf{k}}$ in detail

$$\begin{aligned} R_{\mathbf{k}} = \tfrac{1}{2}(2\pi)^{-6}\int &\{(\mathbf{k}+\mathbf{j})^2[1-2\cos^2\theta_{kj}\cos^2\theta_{kl}+\cos\theta_{kj}\cos\theta_{kl}\cos\theta_{jl}] \\ &-(\mathbf{k}^2+\mathbf{j}^2+\mathbf{k}.\mathbf{j})[\cos^2\theta_{kl}-\cos^2\theta_{jl}]\}\,\delta(\mathbf{k}+\mathbf{j}+\mathbf{l}) \\ &\times q_{\mathbf{j}}(R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}})\,d^3j\,d^3l \end{aligned} \tag{8.48}$$

$$= \tfrac{1}{2}(2\pi)^{-6}\int[\mathbf{k}^2(1-\cos^2\theta_{kj})+p_{\mathbf{kj}}]\,q_{\mathbf{j}}(R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}})^{-1}\,d^3j, \tag{8.49}$$

where $p_{\mathbf{kj}}$ tends to zero with $|\mathbf{j}|$. Suppose that for $|\mathbf{j}| < J$ the value of $q_{\mathbf{j}}$ directly mirrors the input, and remove this region from the integral. Near $\mathbf{j} = 0$, $R_{\mathbf{j}}$ is small so one may write

$$R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}} \cong 2R_{\mathbf{k}} \tag{8.50}$$

giving

$$R_{\mathbf{k}} = \frac{\mathbf{k}^2}{R_{\mathbf{k}}}\mathscr{A}+\int_{|\mathbf{j}|>J}\frac{\mathbf{k}^2(1-\cos\theta_{kj})+p_{\mathbf{kj}}}{R_{\mathbf{k}}+R_{\mathbf{j}}+R_{-\mathbf{k}-\mathbf{j}}}\,q_{\mathbf{j}}\,d^3j.$$

If again $R_{\mathbf{k}}$ and $q_{\mathbf{k}}$ are taken tentatively to be powers, the integral as before can be transformed into

$$\frac{|\mathbf{k}|^5 q_{\mathbf{k}}}{R_{\mathbf{k}}}\int_{J/|\mathbf{k}|<|\mathbf{j}|}\frac{[(1-\cos^2\theta_{kj})+p_{\mathbf{kj}}|\mathbf{k}|^{-2}]\,|\mathbf{j}|^{-m}\,d^3j}{1+|\mathbf{j}|^n+|(\mathbf{1}+\mathbf{j})|^n}. \tag{8.51}$$

This integral will not be independent of $\mathbf{k}$ unless it converges as $J \to 0$. This is the case for the part depending upon $p_{\mathbf{kj}}$ but not for the first term which again gives a contribution like $\mathbf{k}^2 R_{\mathbf{k}}^{-1}$. So altogether one may write

$$R_{\mathbf{k}} \sim \mathscr{B}\mathbf{k}^2 R_{\mathbf{k}}^{-1} + \mathscr{C}|\mathbf{k}|^5 q_{\mathbf{k}} R_{\mathbf{k}}^{-1}. \tag{8.52}$$

This now contradicts the original assumption of a pure power law since it suggests that R lies between

$$R_{\mathbf{k}} \propto |\mathbf{k}| \tag{8.53}$$

and

$$R_{\mathbf{k}} \propto |\mathbf{k}|^{\frac{5}{2}} q_{\mathbf{k}}^{\frac{1}{2}}. \tag{8.54}$$

Assuming the former, since $8 = 2m+n$, one has

$$q_{\mathbf{k}} \propto |\mathbf{k}|^{-\frac{7}{2}}, \quad R_{\mathbf{k}} \propto |\mathbf{k}|. \tag{8.55}$$

This law has been obtained by Kraichnan in the paper mentioned earlier. Kraichnan gives a discussion of the experimental situation in this paper. If one adopts the other extreme, one obtains the Kolmogoroff result

$$q_{\mathbf{k}} \propto |\mathbf{k}|^{-\frac{11}{3}}.$$

Presumably the complete solution will lie between these two extremes which are in fact very close to one another. These remarks again apply to all higher order terms which have the effect of modifying the coefficients $\mathscr{B}$ and $\mathscr{C}$. The Kolmogoroff hypothesis in the present context is that the coefficient $\mathscr{B}$ is negligible compared to $\mathscr{C}$, but there seems no reason in the present analysis for this to be the case. The surprising point is that if one makes the exact opposite of the Kolmogoroff assumption: that the energy input into a component of large k is directly dependent on the behaviour of the system in the external input region, i.e. $\mathscr{B} \gg \mathscr{C}$, one only changes the $|\mathbf{k}|$ dependence of $q_{\mathbf{k}}$ from $|\mathbf{k}|^{-\frac{22}{6}}$ to $|\mathbf{k}|^{-\frac{21}{6}}$.

There now remains the time dependence to be investigated and this will be done in the next section.

9. The time correlation of the velocity correlation functions

The basic equation

$$(ik_0 - \omega_{\mathbf{k}})\,\mathscr{Q}_k = (ik_0 - \omega_{\mathbf{k}})^{-1} h_{\mathbf{k}} + \mathscr{S}_k \tag{9.1}$$

may be integrated over all k_0, and in order that it will reproduce the equations of § 6 one must have

$$\int k_0 \mathscr{Q}_k \, dk_0 = 0, \tag{9.2}$$

i.e.

$$\left.\frac{\partial \mathscr{Q}_{\mathbf{k}}(t)}{\partial t}\right|_{t=0} = 0. \tag{9.3}$$

In time-dependent form one has then to solve

$$\left(\frac{\partial}{\partial t} + \omega_{\mathbf{k}}\right) Q_{\mathbf{k}}(t)\, q_{\mathbf{k}} = h_{\mathbf{k}} e^{-\omega_{\mathbf{k}} t} + \mathscr{S}_{\mathbf{k}}(t), \tag{9.4}$$

with the boundary conditions

$$\left.\begin{aligned} Q_{\mathbf{k}}(0) &= 1, \\ \dot{Q}_{\mathbf{k}}(t)|_{t=0} &= 0, \end{aligned}\right\} \tag{9.5}$$

which will assure that

$$\mathscr{S}_{\mathbf{k}}(0) = S_{\mathbf{k}}, \tag{9.6}$$

and so $\omega_{\mathbf{k}} q_{\mathbf{k}} = h_{\mathbf{k}} + S_{\mathbf{k}}$. The time dependence of $\mathscr{S}_k$ is highly involved since it has a highly convoluted structure. If one tentatively associated a time dependence of $\exp(-\omega_{\mathbf{k}} t)$ with $Q_{\mathbf{k}}(t)$ then $\mathscr{S}_{\mathbf{k}}$ would behave as an average over

$$\exp[-(\omega_{\mathbf{k}} + \omega_{-\mathbf{k}-\mathbf{j}})\, t].$$

In different regions of $\mathbf{k}$ space $(\omega_{\mathbf{k}} + \omega_{-\mathbf{k}-\mathbf{j}})^{-1}$ will be larger or smaller than $\omega_{\mathbf{k}}^{-1}$. In particular, if $\omega_{\mathbf{k}}$ is a power, $(\omega_{\mathbf{k}} + \omega_{-\mathbf{k}-\mathbf{j}})^{-1}$ will be a maximum for $\mathbf{j} = \frac{1}{2}\mathbf{k}$. Though it is now clear that $Q_{\mathbf{k}}$ cannot have a simple exponential decay one can still argue that $\mathscr{S}_{\mathbf{k}}$ will consist of some part decaying slower than $Q_{\mathbf{k}}$ and some part faster, so a crude assessment of the situation will be to write

$$\mathscr{S}_{\mathbf{k}} = W_{\mathbf{k}} + V_{\mathbf{k}}, \tag{9.7}$$

where $W_{\mathbf{k}}$ decays faster than $Q_{\mathbf{k}}$, $V_{\mathbf{k}}$ slower. For short times, the system moves slowly because $\mathscr{Q} = 0$ at $t = 0$. This means that initially

$$\mathscr{Q}_{\mathbf{k}} = q_{\mathbf{k}} - \beta_{\mathbf{k}} t^2, \tag{9.8}$$

i.e.

$$\mathscr{Q}_{\mathbf{k}} = q_{\mathbf{k}} \exp(-\epsilon_{\mathbf{k}} t^2). \tag{9.9}$$

After a while $W_{\mathbf{k}}$ will become small, $V_{\mathbf{k}}$ will still be slowly varying, so the behaviour at intermediate times will follow the solution of

$$\left(\frac{\partial}{\partial t} + \omega_{\mathbf{k}}\right) \mathscr{Q}_{\mathbf{k}}(t) = w_{\mathbf{k}} \delta(t) + V_{\mathbf{k}}(t), \tag{9.10}$$

i.e.

$$\mathscr{Q}_{\mathbf{k}}(t) = p_{\mathbf{k}}(t)\, e^{-\omega_{\mathbf{k}} t}, \tag{9.11}$$

where $p_{\mathbf{k}}(t)$ is some slowly varying function. Finally, at very long times only the most slowly varying components remain, and following the suggestion above a crude model will be to consider

$$V_{\mathbf{k}} = v_{\mathbf{k}} \mathscr{Q}^2_{\frac{1}{2}\mathbf{k}} \tag{9.12}$$

and since $\dot{\mathscr{Q}}_{\mathbf{k}}$ is small, therefore

$$\omega_{\mathbf{k}} \mathscr{Q}_{\mathbf{k}} = v_{\mathbf{k}} \mathscr{Q}^2_{\frac{1}{2}\mathbf{k}}. \tag{9.13}$$

This equation has the solution $(v_{\mathbf{k}}/\omega_{\mathbf{k}})\, t^{-\gamma|\mathbf{k}|}$ where γ is an arbitrary constant chosen to fit on to the intermediate solution. It is very crude of course to assume that all $\mathscr{Q}_{\mathbf{j}}$ decaying more slowly than $\mathscr{Q}_{\mathbf{k}}$ do so at the minimum rate, but a more elaborate argument allowing for the variation of $\mathscr{Q}_{\mathbf{j}}$ leads to a time dependence

$$(\log t)^{\frac{3}{2}}\, t^{-\gamma|\mathbf{k}|}$$

which is not very different on a logarithmic scale. These arguments can again be applied to higher terms in the expansion and rather surprisingly still go through. For example, one finds corrections to (9.13) of the type $\mathscr{Q}^3_{\frac{1}{3}\mathbf{k}}$ and so on, which again leads to the final power law, so the general picture of an initial Gaussian form (9.9), followed by a main exponential region (9.11) characterized by $\omega_{\mathbf{k}}$, then finally a power law tail is independent of the order of the approximation. There is some correlation between any one $u_{\mathbf{k}}$ and any other $u_{\mathbf{j}}$, and between any $u_{\mathbf{k}}$ and the input. For very long times in the behaviour of every $u_{\mathbf{j}}$ will be found the residue of the behaviour of the most slowly varying parts of the system, either of those $u_{\mathbf{j}}$ for $\mathbf{j} \sim 0$, or of the input should there be slowly varying components in it (which have not been considered here).

10. The accuracy of the expansion

In much of the work of §§ 8 and 9 it has been possible to state that the higher terms of the expansion do not alter the functional form of the solution but only the constants which appear. There still remains the question of how well these constants are represented by the simplest approximations to S and R. In the case of an input $h(|\mathbf{k}|/k_1)^{-\alpha}$ it has been shown that the solution can be written in terms of the two constants ρ_α, σ_α of (8.30), the coefficients of turbulent viscosity and turbulent diffusion. The series for $S_\mathbf{k}$ and $R_\mathbf{k}$ thus reduce to a pair of equations of the type

$$\sigma_\alpha = \sigma^{(1)}(\sigma_\alpha, \rho_\alpha) + \sigma^{(2)}(\sigma_\alpha, \rho_\alpha) + \ldots,$$
$$\rho_\alpha = \rho^{(1)}(\sigma_\alpha, \rho_\alpha) + \rho^{(2)}(\sigma_\alpha, \rho_\alpha) + \ldots.$$

There is no functional problem remaining and in principle the solution of these equations can be found numerically. But the integrals involved are considerably more complicated than say the radiative corrections to the Lamb shift which are the most difficult integrals of this type to have been performed, to the author's knowledge. So the best that can be offered at this stage is a discussion of the features of these integrals which will lead one to suppose them to be small. As has been noted, whereas $S_\mathbf{k}$ and $R_\mathbf{k}$ in the first approximation are both given by single positive integrals, the higher approximations are mixtures of positive and negative terms, as many positive as negative in both S and R. This fact is connected with the existence of the exactly soluble model (8.13), and since the integrations are heavily entwined in all the higher terms one may hope that the positive and negative integrals will approximately cancel. It is possible to go a little further with the power input model if the integrations of say fourth-order terms are performed approximately by altering the denomination in such a way that the resulting integrals are products of those for S and R. This can be done without spoiling the feature of equal positive and negative terms, and then one finds that the correction to $q_\mathbf{k}$ due to the fourth-order terms is of order

$$(S_\mathbf{k} - R_\mathbf{k} q_\mathbf{k})^2 / R_\mathbf{k} q_\mathbf{k},$$

i.e.

$$q_\mathbf{k} = d_\mathbf{k} R_\mathbf{k}^{-1} + O(S_\mathbf{k} - R_\mathbf{k} q_\mathbf{k})^2 R_\mathbf{k}^{-1} q_\mathbf{k}^{-1} \tag{10.1}$$

$$= \frac{d_\mathbf{k}}{R_\mathbf{k}}\left[1 + O\left(\frac{h_\mathbf{k}}{d_\mathbf{k}}\right)^2\right]. \tag{10.2}$$

(The precise statement of the approximation is given in Appendix 4 with the discussion of higher order terms.) The expansion in this approximation is then in the ratio of the external input squared to the internal input squared, and the accuracy of the expansion is measured by the smallness of the external input required to keep the system steady. This argument is of course limited to the power input case and does not mean that if $h_\mathbf{k} = 0$ anywhere the method is exact in that region of $\mathbf{k}$ space.

Since these purely mathematical arguments are not very convincing it is worth considering the situation from the physical point of view and asking what phenomena are considered small by stopping at the first approximations for $S_\mathbf{k}$ and $R_\mathbf{k}$. The distribution function

$$F = F_0 + F_1 + F_2 + \ldots$$

predicts the correlation of three velocities

$$\langle U^{\alpha}_{\mathbf{k}} U^{\beta}_{\mathbf{j}} U^{\gamma}_{\mathbf{l}} \rangle = \delta_{\mathbf{k}+\mathbf{j}+\mathbf{l}} \sum_{\substack{\alpha'\beta'\gamma' \\ \text{and permutations}}} \frac{\mathscr{D}^{\alpha\alpha'}_{\mathbf{k}} \mathscr{D}^{\beta\beta'}_{\mathbf{j}} \mathscr{D}^{\gamma\gamma'}_{\mathbf{l}} q_{\mathbf{l}} q_{\mathbf{j}} M^{\alpha'\beta'\gamma'}_{\mathbf{kjl}} \Delta}{\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}}}, \tag{10.3}$$

but it also predicts correlations of four and six velocities which cannot be expressed as simple factors

$$\begin{aligned} \langle U^{\alpha}_{\mathbf{k}} U^{\beta}_{\mathbf{j}} U^{\gamma}_{\mathbf{l}} U^{\epsilon}_{\mathbf{n}} \rangle = {} & \text{(terms which vanish if all } \mathbf{k}, \mathbf{j}, \mathbf{l}, \mathbf{m} \text{ are unequal)} \\ & + \sum_{\substack{\alpha'\beta'\gamma'\alpha''\beta''\gamma'' \\ \mathbf{k}'\mathbf{j}'\mathbf{l}'\mathbf{k}''\mathbf{j}''\mathbf{l}'' \\ \text{permutations}}} M^{\alpha'\beta'\gamma'}_{\mathbf{k}'\mathbf{j}'\mathbf{l}'} M^{\alpha''\beta''\gamma''}_{\mathbf{k}''\mathbf{j}''\mathbf{l}''} \mathscr{D}^{\alpha\alpha'}_{\mathbf{k}} \mathscr{D}^{\beta\gamma''}_{\mathbf{j}} \mathscr{D}^{\gamma\gamma'}_{\mathbf{j}} \mathscr{D}^{\delta\alpha''}_{\mathbf{n}} \mathscr{D}^{\beta''\beta'}_{\mathbf{j}'} \Delta^2 \\ & \quad \times q_{\mathbf{j}} q_{\mathbf{j}'} q_{\mathbf{l}} \delta_{\mathbf{k}+\mathbf{k}''} \delta_{\mathbf{j}'-\mathbf{j}''} \delta_{\mathbf{l}+\mathbf{l}'} \delta_{\mathbf{n}+\mathbf{k}''} \delta_{\mathbf{j}-\mathbf{l}''} \delta_{\mathbf{n}-\mathbf{k}''} \\ & \quad \times (\omega_{\mathbf{k}''} + \omega_{\mathbf{j}''} + \omega_{\mathbf{l}''})^{-1} (\omega_{\mathbf{l}'} + \omega_{\mathbf{k}'} + \omega_{\mathbf{l}''} + \omega_{\mathbf{k}''})^{-1} \\ & + \sum_{\substack{\alpha'\beta'\gamma'\alpha''\beta''\gamma'' \\ \mathbf{k}'\mathbf{j}'\mathbf{l}'\mathbf{k}''\mathbf{j}''\mathbf{l}'' \\ \text{permutations}}} M^{\alpha'\beta'\gamma'}_{\mathbf{k}'\mathbf{j}'\mathbf{l}'} M^{\alpha''\beta''\gamma''}_{\mathbf{k}''\mathbf{j}''\mathbf{l}''} \mathscr{D}^{\alpha\alpha'}_{\mathbf{k}} \mathscr{D}^{\beta\beta''}_{\mathbf{j}} \mathscr{D}^{\gamma\gamma'}_{\mathbf{l}} \mathscr{D}^{\epsilon\gamma''}_{\mathbf{n}} \mathscr{D}^{\beta'\alpha''}_{\mathbf{j}'} \Delta^2 \\ & \quad \times q_{\mathbf{l}} q_{\mathbf{j}} q_{\mathbf{n}} \delta_{\mathbf{j}'+\mathbf{k}'} \delta_{\mathbf{k}+\mathbf{k}'} \delta_{\mathbf{j}+\mathbf{j}''} \delta_{\mathbf{l}+\mathbf{l}'} \delta_{\mathbf{n}+\mathbf{l}''} \\ & \quad \times (\omega_{\mathbf{k}''} + \omega_{\mathbf{j}''} + \omega_{\mathbf{l}''})^{-1} (\omega_{\mathbf{l}'} + \omega_{\mathbf{k}'} + \omega_{\mathbf{j}''} + \omega_{\mathbf{l}''})^{-1}, \end{aligned} \tag{10.4}$$

$$\begin{aligned} \langle U^{\alpha}_{\mathbf{k}} U^{\beta}_{\mathbf{j}} U^{\gamma}_{\mathbf{l}} U^{\epsilon}_{\mathbf{m}} U^{\lambda}_{\mathbf{n}} U^{\mu}_{\mathbf{p}} \rangle = {} & \text{(terms which vanish if } \mathbf{k} \ldots \mathbf{p} \text{ are not all unequal)} \\ & + \sum_{\substack{\alpha'\beta'\gamma'\alpha''\beta''\gamma'' \\ \mathbf{k}'\mathbf{j}'\mathbf{l}'\mathbf{k}''\mathbf{j}''\mathbf{l}'' \\ \text{permutations}}} M^{\alpha'\beta'\gamma'}_{\mathbf{k}'\mathbf{j}'\mathbf{l}'} M^{\alpha''\beta''\gamma''}_{\mathbf{k}''\mathbf{j}''\mathbf{l}''} \mathscr{D}^{\alpha\alpha'}_{\mathbf{k}} \mathscr{D}^{\beta\beta'}_{\mathbf{j}} \mathscr{D}^{\gamma\gamma'}_{\mathbf{l}} \mathscr{D}^{\epsilon\alpha''}_{\mathbf{m}} \mathscr{D}^{\lambda\beta''}_{\mathbf{n}} \mathscr{D}^{\mu\gamma''}_{\mathbf{p}} \Delta^2 \\ & \quad \times q_{\mathbf{j}} q_{\mathbf{l}} q_{\mathbf{n}} q_{\mathbf{p}} \delta_{\mathbf{k}+\mathbf{k}'} \delta_{\mathbf{j}+\mathbf{j}'} \delta_{\mathbf{l}+\mathbf{l}'} \delta_{\mathbf{m}+\mathbf{k}''} \delta_{\mathbf{n}+\mathbf{j}''} \delta_{\mathbf{p}+\mathbf{l}''} \\ & \quad \times (\omega_{\mathbf{k}''} + \omega_{\mathbf{j}''} + \omega_{\mathbf{l}''})^{-1} (\omega_{\mathbf{k}'} + \omega_{\mathbf{j}'} + \omega_{\mathbf{l}'} + \omega_{\mathbf{k}''} + \omega_{\mathbf{j}''} + \omega_{\mathbf{l}''})^{-1}. \end{aligned} \tag{10.5}$$

To ignore F_5, F_6, etc., is to assume that the five U correlation can be expressed in terms of the three and the two, and the eight U correlation can be expressed in terms of the six and the two, and the extent to which they are not is a measure of the inaccuracy of the theory. This is rather a remote physical effect. A much simpler one is to consider the probability that the velocity at a point $\mathbf{x}$ is $\mathbf{U}$. This is given by

$$f(\mathbf{U}) = \int \delta(\mathbf{u}(\mathbf{x}) - \mathbf{U}) F \Pi \, du \tag{10.6}$$

and will be independent of $\mathbf{x}$ in the homogeneous case. Clearly if the expansion for F is used it develops f in a series of Hermite polynomials

$$\int F_i \delta(\mathbf{u}(\mathbf{x}) - \mathbf{U}) \, \Pi \, du = f_0 \sum_n a_i^{(n)} H_n(U). \tag{10.7}$$

The basic Gaussian f_0 is chosen so that H_2 never appears. Working to second order, since by symmetry $a_1^{(n)}$ is zero, one has fourth- and sixth-order polynomials alone

$$\begin{aligned} f &= f_0 + f_2, \\ f_2 &= f_0(1 + aH_4 + bH_6). \end{aligned} \tag{10.8}$$

The residual terms in F_2 are the cause of the subsequent terms of the expansion, so one may now say that the expansion should be good if f is a Gaussian with small additional terms in H_4, H_6 and no further corrections such as the H_8, H_{10}, H_{12} which arise from F_4. A more detailed analysis shows that the effect of the F_2 corrections is to make the Gaussian rather more peaked at the origin.

More elaborate correlation functions such as the joint probability of finding $\mathbf{U}_1$ at $\mathbf{X}_1$ while $\mathbf{U}_2$ at $\mathbf{X}_2$ can be defined

$$f(\mathbf{U}_1, \mathbf{X}_1;\ \mathbf{U}_2, \mathbf{X}_2) = \int \delta(\mathbf{u}(\mathbf{X}_1) - \mathbf{U}_1)\, \delta(\mathbf{u}(\mathbf{X}_2) - \mathbf{U}_2)\, F \Pi\, du, \tag{10.9}$$

and discussed in a similar fashion.

Similar arguments can be applied to the time-dependent case. It will be noted that $q_{\mathbf{k}}$, $R_{\mathbf{k}}$ and $h_{\mathbf{k}}$ form a soluble set of functions without discussing time dependence so that if they could be obtained from experiment, the equation (7.16) governing time dependence could be viewed as one for $Q_{\mathbf{k}}$ alone with $q_{\mathbf{k}}$, $R_{\mathbf{k}}$, $h_{\mathbf{k}}$ put in as externally defined functions.

11. Conclusions

Many problems in theoretical physics can be expressed in terms of functional differential equations, but turbulence is an exceptional problem in that there is in the limit of large Reynolds number no external parameter which can be used as a basis of an expansion technique. In the language of quantum field theory it is a problem of infinitely strong coupling constant. It follows that an expansion must be based on the internal properties of the system and with one's present limited knowledge of non-trivial mathematical operations in Hilbert space the only substantial fact is that since the probability of finding a particular velocity at a particular point in a fluid is quite close to a Gaussian (Batchelor 1959, ch. 8), the system is substantially random and the generalized random phase approximation should be applicable. This method appears to be the simplest which stems directly from Liouville's equation or the generalized phase space equation, and is entirely consistent to any order in the sense that it contains no features which contradict the original equations from which it was derived. The situations discussed in this paper are all highly idealized, being, it is believed, as simple as can be whilst still containing the mathematical essence of the problem. For this reason no detailed comparisons with experiments are offered, though it is hoped that the method of attack will prove a sound basis for the discussion of real situations and further calculations to this end are in hand.

The author would like to thank Dr Batchelor for a helpful discussion which led to an expansion of § 8, Dr Kraichnan for a correspondence which eliminated several errors from the first draft, and caused the author to invent the work of § 3. The first draft was an account of lectures given by the author at the Culham Laboratory in 1961, and the author is grateful for the opportunity to visit the laboratory on that and many other occasions and for many discussions there, particularly with Prof. W. B. Thompson. He would like also to thank Prof. Flowers for many helpful discussions concerning the problems of turbulence and the writing of this paper.

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Appendix 1

The equation of motion
$$\dot{U} + JU = \mathscr{F}(t) \tag{A 1.1}$$
can be solved directly with the boundary condition $U(t') = u'$, giving
$$U = u' \exp[-J(t-t')] + \int_{t'}^{t} \exp[-J(t-\tau)]\mathscr{F}(\tau)\,d\tau. \tag{A 1.2}$$
The Green function for the motion is then
$$G = \delta\left(u - u'\exp[-J(t-t')] - \int_{t'}^{t}\exp[-J(t-\tau)]\mathscr{F}(\tau)\,d\tau\,\Theta(t-t')\right). \tag{A 1.3}$$
It follows that the mean Green function is given by
$$\langle G\rangle = \mathscr{N}\int \delta\mathscr{F}\exp\left[-\frac{1}{2}\int_{t'}^{t}\int_{t'}^{t}\mathscr{F}(\tau)\,g^{-1}(\tau-\tau')\,\mathscr{F}(\tau')\,d\tau\,d\tau'\right]G. \tag{A 1.4}$$
Write G parametrically by using the integral representation of the δ function
$$G = (2\pi)^{-1}\Theta\int_{-\infty}^{\infty} d\lambda \exp i\lambda\left(u - u'\exp[-J(t-t')] - \int_{t'}^{t}\exp[-J(t-\tau)]\mathscr{F}(\tau)\,d\tau\right). \tag{A 1.5}$$
By substituting in (2.17) the integration over $\mathscr{F}$ is performed by completing the square, i.e. by changing variables to
$$\mathscr{F}'(\tau) = \mathscr{F}(\tau) - i\lambda\int_{t'}^{t} g(\tau-\tau')\exp[-J(t-\tau')]\,d\tau'\,\Theta(t-\tau)\,\Theta(\tau-t'), \tag{A 1.6}$$
which leaves
$$\langle G\rangle = (2\pi)^{-1}\Theta\int_{-\infty}^{\infty} d\lambda\exp\left\{i\lambda(u-u'\exp[-J(t-t')]) - \lambda^2\int_{t'}^{t}\int_{t'}^{t}\exp[-J(t-\tau)-J(t-\tau')]\,g(\tau-\tau')\,d\tau\,d\tau'\right\}. \tag{A 1.7}$$
This is finally evaluated by again completing the square to give
$$\langle G\rangle = (I/\pi)^{\frac{1}{2}}\exp[-(u-u'e^{-J(t-t')})^2/I(t,t')] \tag{A 1.8}$$
where
$$I(t,t') = \frac{1}{2}\int_{t'}^{t}\int_{t'}^{t}\exp[-J(t-\tau_1)-J(t-\tau_2)]\,g(\tau_1-\tau_2)\,d\tau_1\,d\tau_2. \tag{A 1.9}$$

Appendix 2

If the non-linear term is added one may still write the solution of

$$\frac{\partial U_{\mathbf{k}}^{\alpha}}{\partial t} = -\nu \mathbf{k}^2 U_{\mathbf{k}}^{\alpha} + \sum_{\beta\gamma;\,\mathbf{jl}} M_{-\mathbf{kjl}}^{\alpha\beta\gamma} U_{\mathbf{j}}^{\beta} U_{\mathbf{l}}^{\gamma} + \sum_{\beta} \mathscr{F}_{\mathbf{k}}^{\beta} \mathscr{D}_{\mathbf{k}}^{\alpha\beta} \tag{A 2.1}$$

as $$U_{\mathbf{k}}^{\alpha} = u_{\mathbf{k}}^{\alpha'} \exp[-\nu \mathbf{k}^2(t-t')] + \int_{t'}^{t} \exp[-\nu \mathbf{k}^2(t-\tau)]\{(MUU)_{\tau} + (\mathscr{F}\mathscr{D})_{\tau}\},$$

and proceeding as before eventually obtain

$$\langle G \rangle = \prod_{\mathbf{k}} (I_{\mathbf{k}}/\pi) \exp\left[-\sum_{\mathbf{k}} \Delta \left(u_{\mathbf{k}}^{\alpha} - u_{\mathbf{k}}^{\alpha'} e^{-\nu \mathbf{k}^2(t-t')} + \int_{t'}^{t} (MUU)_{\tau}^{\alpha} e^{-\nu \mathbf{k}^2(t-\tau)} d\tau\right)^2\right], \tag{A 2.2}$$

with the condition $\operatorname{div} \mathbf{u} = 0$.

This of course is of no use as it stands since the unknown interaction term remains, but upon explicitly differentiating $\langle G \rangle$, as in (4.13) above, the interaction term only appears at the time t, when $\mathbf{U}$ is just $\mathbf{u}$, so one still has the simple form

$$\left(\frac{\partial}{\partial t} + \sum_{\alpha,\mathbf{k}} \frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}} \left(\sum_{\beta} \mathscr{D}_{\mathbf{k}}^{\alpha\beta} \Delta^{-1} \frac{\partial}{\partial u_{-\mathbf{k}}^{\beta}} + \nu \mathbf{k}^2 u_{\mathbf{k}}^{\alpha} - \sum_{\mathbf{jl};\,\beta\gamma} M_{-\mathbf{kjl}}^{\alpha\beta\gamma} u_{\mathbf{j}}^{\beta} u_{\mathbf{l}}^{\gamma}\right)\right) \langle G \rangle = \delta(t-t') \prod_{\mathbf{k}} \delta(u_{\mathbf{k}} - u_{\mathbf{k}'}) \tag{A 2.3}$$

and the similar equation for $\langle F \rangle$.

Appendix 3

One needs to evaluate the expression

$$M_{\mathbf{kjl}}^{\alpha\beta\gamma} M_{-\mathbf{kj}'\mathbf{l}'}^{\alpha'\beta'\gamma'} (\mathscr{D}_{\mathbf{j}}^{\beta\beta'} \mathscr{D}_{\mathbf{l}}^{\gamma\gamma'} + \mathscr{D}_{\mathbf{j}}^{\beta\gamma'} \mathscr{D}_{\mathbf{l}}^{\gamma\beta'}) \tag{A 3.1}$$

to obtain the coefficient in the integral for S.

Now $S_{\mathbf{k}}^{\alpha\alpha'}$ must have the structure $\mathscr{D}_{\mathbf{k}}^{\alpha\alpha'} S_{\mathbf{k}}$ so to simplify the analysis since

$$\sum_{\alpha} \mathscr{D}_{\mathbf{k}}^{\alpha\alpha} = 2 \tag{A 3.2}$$

one may write the coefficient as

$$(2\pi)^{-6} \tfrac{1}{8} (k^{\beta} \mathscr{D}_{j}^{\alpha\gamma} + k^{\gamma} \mathscr{D}_{k}^{\alpha\beta})(k^{\beta} \mathscr{D}_{k}^{\alpha\gamma'} + k^{\gamma'} \mathscr{D}_{k}^{\alpha\beta'})(\mathscr{D}_{j}^{\beta\beta'} \mathscr{D}_{l}^{\gamma\gamma'} + \mathscr{D}_{j}^{\beta\gamma'} \mathscr{D}_{l}^{\gamma\beta'}). \tag{A 3.3}$$

In a compressed notation this is

$$\begin{aligned}
&(2\pi)^{-6} \tfrac{1}{8} \{2(\mathbf{k}\mathscr{D}_{\mathbf{j}}\mathbf{k})(\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{l}}) + 2(\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathbf{k})(\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{j}}) + 2(\mathbf{k}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{l}}\mathbf{k}) + 2(\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{j}}\mathbf{k})\} \\
&= (2\pi)^{-6} \tfrac{1}{4} k^2 \{(1-\cos^2\theta_{kj})(1+\cos^2\theta_{kl}) + (1-\cos^2\theta_{kl})(1+\cos^2\theta_{kj}) \\
&\qquad + 2\cos\theta_{kj}\cos\theta_{kl}(\cos\theta_{jl} - \cos\theta_{kj}\cos\theta_{kl})\} \\
&= (2\pi)^{-6} \tfrac{1}{2} k^2 \{1 - 2\cos^2\theta_{kj}\cos^2\theta_{kl} + \cos\theta_{kj}\cos\theta_{kl}\cos\theta_{lj}\}.
\end{aligned} \tag{A 3.4}$$

Similarly for R one has the expression

$$\sum_{\substack{\alpha\beta\gamma \\ \alpha'\beta'\gamma'}} \tfrac{1}{2} M_{\mathbf{kjl}}^{\alpha\beta\gamma} (M_{-\mathbf{l}-\mathbf{k}-\mathbf{j}}^{\beta'\alpha\gamma'} + M_{-\mathbf{l}-\mathbf{j}-\mathbf{k}}^{\beta'\gamma'\alpha})(\mathscr{D}_{\mathbf{j}}^{\beta\beta'} \mathscr{D}_{\mathbf{l}}^{\gamma\gamma'} + \mathscr{D}_{\mathbf{j}}^{\beta\gamma'} \mathscr{D}_{\mathbf{l}}^{\beta'\gamma}) \tag{A 3.5}$$

which when written out becomes

$$(2\pi)^{-6} \tfrac{1}{2} \{(\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{k}}\mathbf{l}) + (\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{j}}\mathbf{l}) + (\mathbf{k}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathbf{l}) + (\mathbf{k}\mathscr{D}_{\mathbf{j}}\mathbf{l})(\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{l}})\}. \tag{A 3.6}$$

This can be written in terms of a symmetric part under the interchange of $\mathbf{k}$ and $\mathbf{j}$

$$(2\pi)^{-6}\tfrac{1}{8}\{(\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{k}}\mathbf{l})+(\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{j}}\mathbf{l})+(\mathbf{k}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathbf{l})+(\mathbf{k}\mathscr{D}_{\mathbf{j}}\mathbf{l})(\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{l}}) \\ +(\mathbf{j}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{j}}\mathbf{l})+(\mathbf{j}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{k}}\mathbf{l})+(\mathbf{j}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{j}}\mathbf{l})+(\mathbf{j}\mathscr{D}_{\mathbf{k}}\mathbf{l})(\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{l}})\} \quad \text{(A 3.7)}$$

and an antisymmetric part

$$(2\pi)^{-6}\tfrac{1}{8}\{(\mathbf{l}\mathscr{D}_{\mathbf{j}}\mathbf{l})+2(\mathbf{k}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{k}}\mathbf{l})-(\mathbf{l}\mathscr{D}_{\mathbf{k}}\mathbf{l})(\mathscr{D}_{\mathbf{j}}\mathscr{D}_{\mathbf{l}})-2(\mathbf{j}\mathscr{D}_{\mathbf{l}}\mathscr{D}_{\mathbf{k}}\mathscr{D}_{\mathbf{j}}\mathbf{l})\}. \quad \text{(A 3.8)}$$

The symmetric part is just the coefficient in S with $\mathbf{l}$ and $\mathbf{k}$ interchanged, whilst the antisymmetric part gives

$$(2\pi)^{-6}\tfrac{1}{2}\{\mathbf{l}^2[\cos^2\theta_{kl}-\cos^2\theta_{jl}]-(\mathbf{k}.\mathbf{j})[\cos^2\theta_{kl}-\cos^2\theta_{jl}]\}. \quad \text{(A 3.9)}$$

In all then, since one may leave the antisymmetric part in $L_{\mathbf{kjl}}$ for convenience in writing, one has

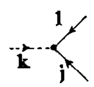
$$L_{\mathbf{kjl}} = \tfrac{1}{2}[(1-2\cos^2\theta_{kj}\cos^2\theta_{kl}+\cos\theta_{kj}\cos\theta_{jl}\cos\theta_{kl})k^2-(\mathbf{k}^2-\mathbf{l}.\mathbf{j})\cos^2\theta_{kj}]. \quad \text{(A 3.10)}$$

Appendix 4

The series is obtained by successively solving equations of the type

$$\sum_{\alpha;\mathbf{k}}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\left(\sum_{\beta}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}\Delta^{-1}\frac{\partial}{\partial u_{-\mathbf{k}}^{\beta}}d_{\mathbf{k}}+\omega_{\mathbf{k}}u_{\mathbf{k}}^{\alpha}\right)F_{n+2} = -\sum_{\alpha\beta\gamma;-\mathbf{kjl}}M_{-\mathbf{kjl}}^{\alpha\beta\gamma}u_{\mathbf{j}}^{\beta}u_{\mathbf{l}}^{\gamma}\,\partial F_{2n+1}/\partial u_{\mathbf{k}}^{\alpha} \\ -\sum_{\alpha;\mathbf{k}}\frac{\partial}{\partial u_{\mathbf{k}}^{\alpha}}\left(\sum_{\beta}\mathscr{D}_{\mathbf{k}}^{\alpha\beta}\Delta^{-1}\frac{\partial}{\partial u_{-\mathbf{k}}^{\beta}}S_{\mathbf{k}}+R_{\mathbf{k}}u_{\mathbf{k}}^{\alpha}\right)F_{2n}. \quad \text{(A 4.1)}$$

To obtain a picture of the right-hand side one must invent a graphical notation (as in the virial cluster expansion of a gas or the Feynman diagrams) for algebraically it becomes very complicated. The diagrams however are quite different from the examples mentioned and are constructed this way. For M write a dot, for u a full line, for $\partial/\partial u$ a dotted line. Then $M_{\mathbf{kjl}}^{\alpha\beta\gamma}u_{\mathbf{j}}^{\beta}u_{\mathbf{l}}^{\gamma}\,\partial/\partial u_{-\mathbf{k}}^{\alpha}$ is written

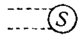


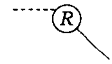
(A 4.2)

where the arrows give a vector sense so that

$$\mathbf{l}+\mathbf{j}+\mathbf{k} = 0.$$

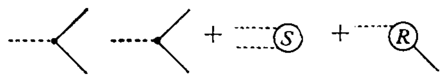
Similarly one can define



and


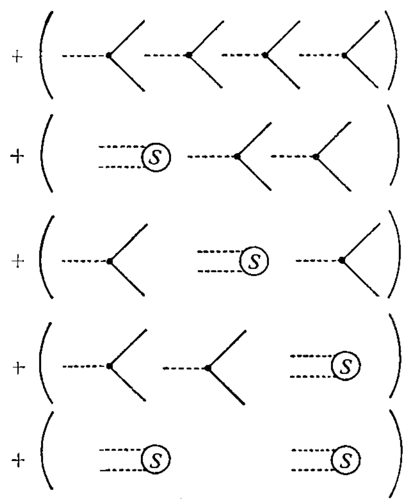
. (A 4.3, A 4.4)

It will be assumed that dotted lines will always be drawn to the left, whilst full lines have arbitrary directions. Now to solve (A 4.1) one needs the right-hand side in Hermite polynomials where upon the inverse of the right-hand differential operator is $(\sum_{\mathbf{k}} n_{\mathbf{k}}\omega_{\mathbf{k}})^{-1}$. It will turn out that these factors can be easily inferred from the diagrams and need not appear explicitly. That being the case F_{n+2} will consist of combinations of the diagrams (A 4.1, 2, 3) running across the paper from right to left as they would in an algebraic expansion. For example F_2 consists of



(A 4.5)

and F_4 of



(A 4.6)

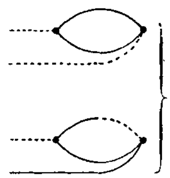
and similar terms in R, and in S and R.

Now the definition of S and R is that

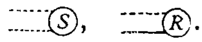
$$\int (F_2 + F_4 + \dots)\, u_{\mathbf{k}} u_{-\mathbf{k}} \Pi\, du = 0. \qquad \text{(A 4.7)}$$

If one starts to perform the integrals it is clear that a $\partial/\partial u_{\mathrm{j}}$ must meet a u_{j} to its left or, by parts, the integral vanishes. In other words the dotted lines in the diagrams either meet and annihilate a full line, or else meet one of the 'external' $u_{\mathbf{k}}$, $u_{-\mathbf{k}}$ of the integral (A 4.7), or else give zero. Clearly they all act to their left. Now consider the remaining full lines. They also give zero unless they can pair with another, i.e. a u_{j} must find another $u_{-\mathrm{j}}$ to be non-vanishing. (This amounts to the same as replacing $u_{\mathrm{j}} u_{-\mathrm{j}}$ by H_0 and $H_{1_{\mathrm{j}} 1_{-\mathrm{j}}}$). In order that a contribution be made to A 4.7, it must follow that all the lines but two in the diagram must link up, two full lines giving a q, a full and a dotted giving unity, or more precisely a $\mathscr{D}$ factor. The two emerging lines consist of the dotted line emerging from the first subdiagram on the left, and then either a dotted line (i.e. S like) or a full line (R like). Both emerging lines are labelled $\mathbf{k}$. It is possible for say $u_{\mathrm{j}} u_{-\mathrm{j}} u_{\mathrm{j}} u_{-\mathrm{j}}$ to appear, i.e. higher Hermite polynomials, but their contribution always turns out to be of order Δ relative to the terms already noted and so the volume will now be assumed so large that this possibility can be ignored. There now remains the terms in $(\Sigma n_{\mathbf{k}} \omega_{\mathbf{k}})^{-1}$. It is clear that the $n_{\mathbf{k}}$ are either zero or unity and they can be characterized this way: the diagrams are well ordered from right to left. So if a vertical line is drawn between each junction it will cut lines of the diagram and for every cut of a line marked j an ω_j is added to the sum and such a factor

$(\Sigma\omega)^{-1}$ is inserted between each junction. Some examples should make this clear. Consider working to order F_2. The diagrams are



(A 4.8)

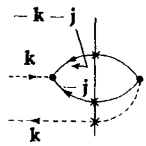


(A 4.9)

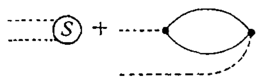
A vertical line gives the factor

$$(\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{-\mathbf{k}-\mathbf{j}})^{-1}$$

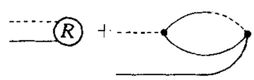
in each of (A 4.8):



The equations of § 5 are got by equating the S and R like parts of A 4.8, 9 to zero,



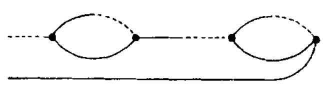
$= 0,$ (A 4.10)



$= 0,$ (A 4.11)

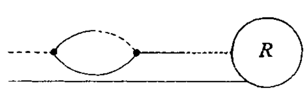
where the two full lines in the 'bubble' in (A 4.10) give $q_{\mathbf{j}}q_{-\mathbf{k}-\mathbf{j}}$, i.e. (5.16), and in (A 4.11) the one full line gives $q_{\mathbf{j}}$, the mixed line $\mathscr{D}$, i.e. (5.19).

If one now goes to fourth order one gets many diagrams. Use the first approximation to S and R, i.e. (A 4.10, 11), some sets of diagrams already completely cancel, for example



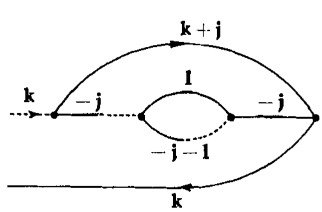
(A 4.12)

cancels exactly with

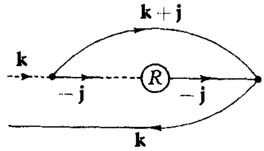


(A 4.13)

Some partially cancel e.g.



(A 4.14)

and 

. (A 4.15)

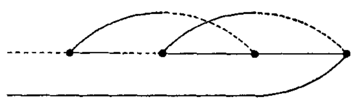
These have the same M and q factors but the $(\Sigma\omega)$ factors are different. For the former, one has

$$(\omega_{-\mathbf{k}-\mathbf{j}}+\omega_{\mathbf{j}}+\omega_{\mathbf{k}})^{-1}(\omega_{-\mathbf{k}-\mathbf{j}}+\omega_{\mathbf{l}}+\omega_{-\mathbf{j}-\mathbf{l}}+\omega_{\mathbf{k}})^{-1}(\omega_{\mathbf{j}}+\omega_{-\mathbf{k}-\mathbf{l}}+\omega_{\mathbf{k}})^{-1},$$

whilst for the latter, recalling the integral for R, one has

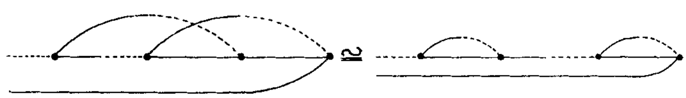
$$(\omega_{-\mathbf{k}-\mathbf{j}}+\omega_{\mathbf{j}}+\omega_{\mathbf{k}})^{-1}(\omega_{\mathbf{l}}+\omega_{-\mathbf{j}-\mathbf{l}}+\omega_{\mathbf{j}})^{-1}(\omega_{\mathbf{j}}+\omega_{-\mathbf{k}-\mathbf{j}}+\omega_{\mathbf{k}})^{-1},$$

differing by a single term in the central factor. (The topologically similar terms in the perturbation expansion of electrodynamics do cancel exactly.) Finally, there are terms which do not contain any subdiagram equivalent to a lower order and are topologically irreducible. Such a term is


(A 4.16)

and with the residue of the partially cancelled terms, these terms give rise to the corrections to R and S in this order. By counting lines and $(\Sigma\omega)^{-1}$ factors, the formal expressions quoted in § 6 are now readily obtained.

The crude evaluation quoted in § 10, is obtained by ignoring the partially cancelled diagrams and assuming the value of the irreducible diagram can be approximated by distorting them into cancelling diagrams, for example


(A 4.17)

Diagrams of the latter type are readily evaluated, the one shown being $R_{\mathbf{k}}^2/\omega_{\mathbf{k}}$. Adding all the types up with due regard to sign one obtains the estimate quoted in § 10.

These diagrammatics can be extended to cover the case in which $\mathscr{P}([\mathscr{F}])$ has a general distribution which is only approximately Gaussian, but since the generalization is quite straightforward it will not be given.